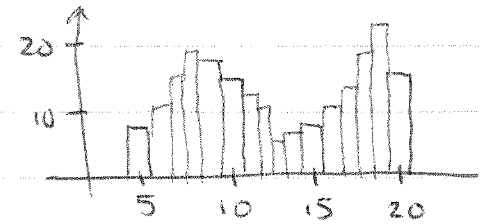
Expectation Values & Operators

* weighted averages.

- ex. average age: $\{5\text{yrs}, 10\text{yrs}, 17\text{yrs}, 20\text{yrs}\}$

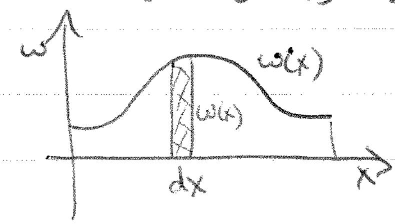
$$\bar{X} \text{ or } \langle x \rangle = (5+10+17+20)\text{yrs}/4 = 13\text{yrs}.$$

- average from histogram $\boxed{n_i}$



$$\langle x \rangle = \frac{n_1 x_1 + n_2 x_2 + \cdots}{\underbrace{n_1}_{\text{weight.}} + n_2 + \cdots}$$

- continuous weight. $\boxed{w(x)}$



$$\langle x \rangle = \frac{\int w(x)dx \cdot x}{\int w(x)dx}$$

- examples: center of mass $= \langle x \rangle_m$

  moment of inertia $= \langle x^2 \rangle_m$

- quantum expectation value: $w(x) = |\psi|^2 = \boxed{\psi^*_{(x)}\,\psi(x)}$

$$\langle x \rangle = \int \psi^*(x) \cdot x \cdot \psi(x)\,dx$$

note : $\int \psi^* \cdot \psi\,dx = 1$ "normalization"

- expectation value of momentum.

$$\frac{\partial}{\partial x} e^{i(kx-\omega t)} = ik\,e^{i(kx-\omega t)} \quad \text{(de Broglie).}$$

so $-i\hbar\frac{\partial}{\partial x} \approx \hbar k = p$

so $\hat{p} \cdot \psi = -i\hbar\frac{\partial}{\partial x}\psi$ ← $\boxed{\text{operators}}$

$$\langle p \rangle = \int \psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx$$

- example: infinite square well. $\psi_n = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ when $0<x<L$ otherwise $\psi = 0$.

* - orthogonality: $\int \psi_n^* \psi_n dx = \frac{2}{L}\int \sin^2\frac{n\pi x}{L} = 1$

- $\int \psi_n^* \psi_m dx = \frac{2}{L}\int \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = 0$ if $n \neq m$.

* <u>list of operators:</u> $\hat{O}$ or $O_{op.}$

time	$\hat{t} = t$	energy	$\hat{E} = i\hbar \frac{\partial}{\partial t}$	kinetic energy	$\hat{T} = \frac{\hat{p}^2}{2m}$
position	$\hat{x} = x$	momentum	$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$	potential energy	$\hat{V} = V(x)$
angle	$\hat{\phi} = \phi$	angular momentum	$\hat{L}_z = p_\phi = -i\hbar \frac{\partial}{\partial \phi}$	Hamiltonian	$\hat{H} = \hat{T} + \hat{V}$

conjugate pairs

* <u>angular momentum:</u>

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\begin{aligned} L_z &= x\, p_y - y\, p_x \\ &= x\left(-i\hbar \frac{\partial}{\partial y}\right) - y\left(-i\hbar \frac{\partial}{\partial x}\right) \\ &= -i\hbar\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \\ &= -i\hbar \frac{\partial}{\partial \phi} \end{aligned}$$

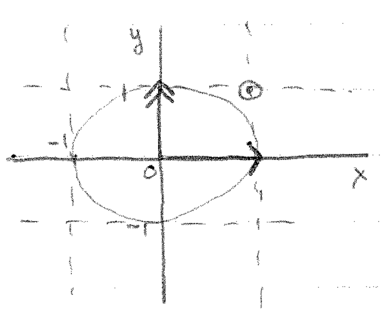
$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$

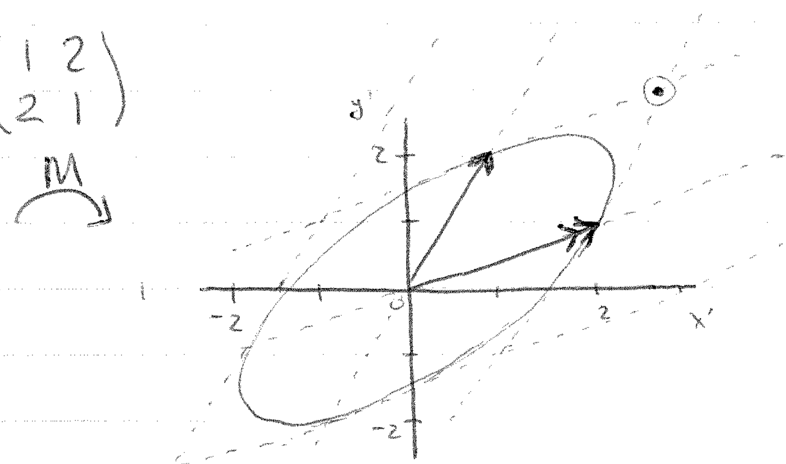
$$\begin{aligned} \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} \\ &= -\rho \sin\phi \frac{\partial}{\partial x} + \rho\cos\phi \frac{\partial}{\partial y} \\ &= x\, \partial/\partial y - y\, \partial/\partial x. \end{aligned}$$

* <u>vector-matrix analogy.</u> (dot product).

unit vectors: $\hat{x}, \hat{y}, \hat{z}$ $\quad \hat{x}\cdot\hat{x} = 1 \quad \hat{x}\cdot\hat{y} = 0$ etc. orthonormality.

matrix operators: $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$





$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \boxed{Mv = \lambda v}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_1 \cdot v_2 = (1\ 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$