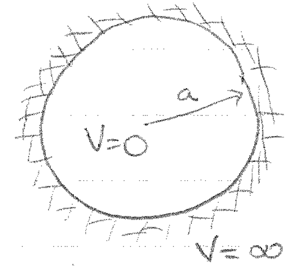
Separation of Variables in Cylindrical Co-ordinates

TISE: $\rho^2 \frac{\partial^2 \psi}{\partial \rho^2} + \rho \frac{\partial \psi}{\partial \rho} - \frac{L_z^2}{\hbar^2}\psi + k^2 \rho^2 \psi = 0, \quad \hbar^2 k^2 = 2m(E-V)$



"infinite square well with circular domain"

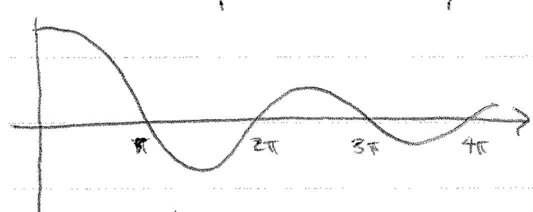
let $\psi(\rho, \phi) = R(\rho) \cdot \Phi(\phi)$

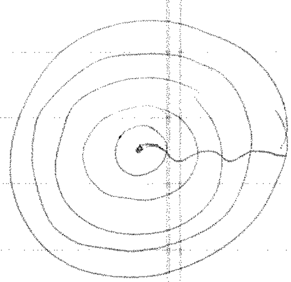
$$\underbrace{\frac{\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + k^2 \rho^2 R}{R}}_{f(r)} = \underbrace{\frac{\frac{\hat{L}_z^2}{\hbar^2}\Phi}{\Phi}}_{f(\phi)} = \underbrace{k_\phi^2}_{\text{constant}}$$

a) if $\Phi(\phi) = 1$ then:

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + k^2 \rho^2 R = 0$$

Bessel functions: $R(\rho) = J_0(k_n \rho)$





since $R(a) = 0 \quad k_n a = 2n\pi \quad k_n = \frac{2n\pi}{a}$

b) solve for the general case of $\Phi(\phi)$.

$$\hat{L}_z^2 \Phi = -\hbar^2 \frac{\partial^2}{\partial \phi^2} \Phi = \hbar^2 k_\phi^2 \Phi \qquad \text{let } \Phi = e^{i k_\phi \phi}$$

$$-i^2 k_\phi^2 \Phi = k_\phi^2 \Phi$$

boundary condition: $e^{i k_\phi \cdot 0} = e^{i k_\phi \cdot 2\pi}$

$$k_\phi = m = 0, 1, 2, \ldots$$

$$\Phi = e^{im\phi} \qquad \boxed{L_z = m\hbar}$$

"Bohr's quantization of angular momentum"

"deBroglie's waves".

* we recover the old quantization condition of the hydrogen atom without even considering the Coulomb potential!

c) full radial solution: $k_\phi = m$ or $L_z = m\hbar$

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + (k^2\rho^2 - m^2) R = 0$$

$R = J_m(k_{nm}\rho)$ also higher order Bessel functions.

boundary condition: $k_{nm} a = x_{nm} = n^{th}$ zero of $J_m(x)$

Abramowitz & Stegun, p. 358-9

BESSEL FUNCTIONS OF INTEGER ORDER 359

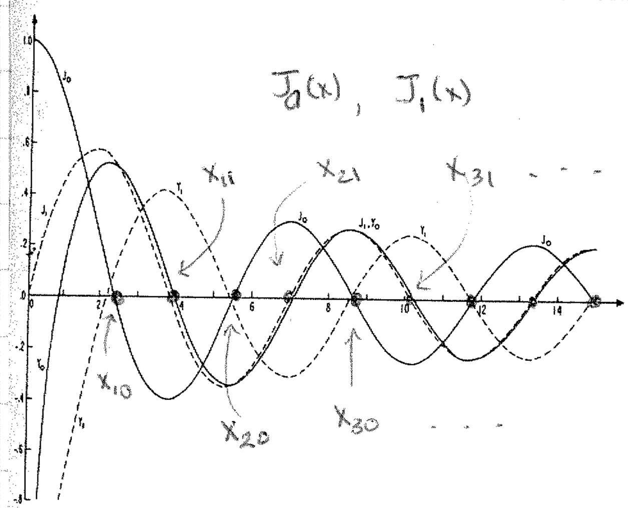


FIGURE 9.1. $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$.

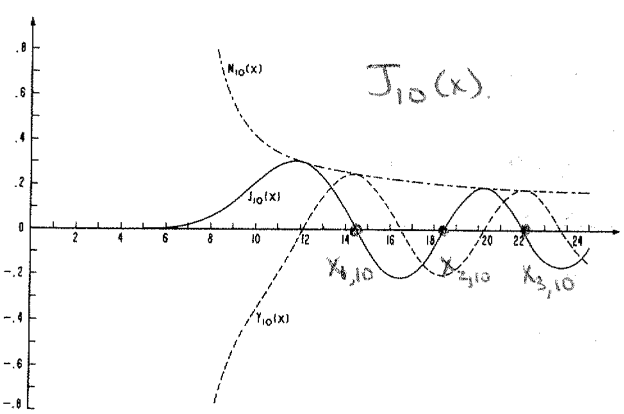


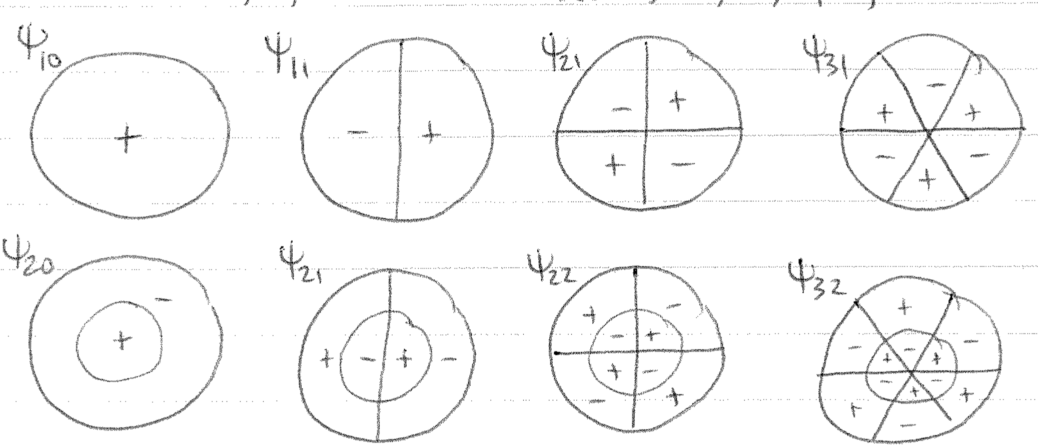
FIGURE 9.2. $J_{10}(x)$, $Y_{10}(x)$, and $M_{10}(x) = \sqrt{J_{10}^2(x) + Y_{10}^2(x)}$.

so $\boxed{\psi_{nm}(\rho,\phi) = A \cdot J_m\left(\underbrace{\frac{x_{nm}\rho}{a}}_{k_{nm}}\right) \cdot e^{im\phi}}$ complex wave function

$$\boxed{E_{nm} = \frac{\hbar^2 x_{nm}^2}{2Ma^2}}$$

quantum number mass

$n = 1, 2, 3 \ldots \quad m = \ldots -2, -1, 0, 1, 2, \ldots$



"node lines" where $\psi = 0$.

"standing waves"

"modes"

"eigenfunctions"