University of Kentucky, Physics 361
Problem Set #4, due Wednesday, Feb 24

1. [30 pts] Complex numbers, circular, and hyperbolic functions:

a) Use the power series of e^x, $\sin(\theta)$, $\cos(\theta)$ to prove Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.

b) Use Euler's identity to show if $z = x + iy$ where (x, y) are cartesian coordinates in the complex plane, then $z = re^{i\theta}$ where (r, θ) are the polar coordinates of the same point.

c) The complex conjugate z^* is formed by replacing i with $-i$ everywhere in z. The modulus $|z| \equiv \sqrt{z^*z}$ is the complex analog of absolute value. Use $z = x + iy = re^{i\theta}$ to show the relations $|z|^2 = z^*z = zz^* = x^2 + y^2 = r^2$. Thus the modulus is the distance of z from the origin.

d) Expand z^2 in terms of x,y and also r,θ to see why $|z|^2$ is more useful in general than z^2.

e) Multiply $e^{i\theta}$ by its complex conjugate and expand using Euler's identity to prove the relation $\sin^2\theta + \cos^2\theta = 1$. This shows that $e^{i\theta}$ traces out a circle in the complex plane.

f) Use Euler's identity on $e^{i\theta}$ and $e^{-i\theta}$ to express $\cos\theta$, $\sin\theta$ and $\tan\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

g) Using the similar definitions $\cosh(\alpha) \equiv \frac{1}{2}(e^{\alpha} + e^{-\alpha})$ and $\sinh(\alpha) \equiv \frac{1}{2}(e^{\alpha} - e^{-\alpha})$, derive the analog of Euler's identity for hyperbolic functions. Hint: i becomes $\pm$.

h) Multiply and expand e^{α} and $e^{-\alpha}$ in two ways to derive a simular formula as in part (e) for $\cosh(\alpha)$ and $\sinh(\alpha)$. This shows that $(\cosh(h), \sinh(\alpha))$ traces out a hyperbola in the plane.

i) Derive the addition formulas for $\cos(\alpha\pm\beta)$ and $\sin(\alpha\pm\beta)$ by multiplying and expanding $e^{i\alpha}\cdot e^{\pm i\beta}$ and then separating the real and imaginary parts.

j) Obtain the derivates of $\sin\theta$ and $\cos\theta$, $\sinh\alpha$, and $\cosh\alpha$ using the derivative of $e^{i\theta}$ and $e^{\pm\alpha}$.

2. [20 pts] Beats and group velocity

a) Show that two waves of equal frequency and amplitude travelling in opposite directions $Ae^{ikx-i\omega t}$ and $Ae^{-ikx-i\omega t}$ superimpose to form a standing wave. What is the resulting wavelength?

b) Using exponentials, show that two pure waves $e^{i(kx-\omega t)}$ of frequency (ω_1, k_1) and (ω_2, k_2), superimpose to form a beat pattern of $2\cos(\Delta k\, x - \Delta\omega\, t)e^{i\bar{k}x-i\bar{\omega}t}$. Derive the formulas for the combined frequencies $(\Delta\omega, \Delta k)$ and $(\bar{\omega}, \bar{k})$. Identify the wave packet and the phase (carrier) wave, and solve the velocity of each. What beat frequency do you hear?

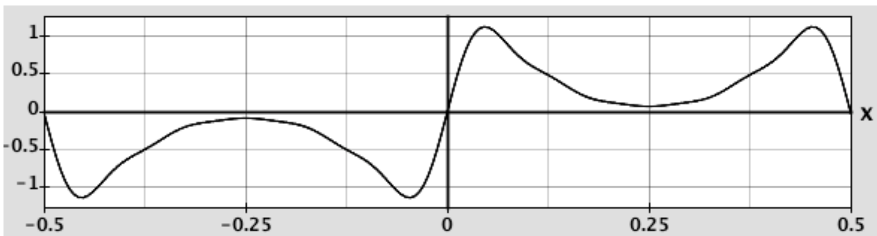
c) Interpret part a) as a specific case of part b).

d) In the limit that the two frequencies are very close together, show that the group velocity is $v_g = d\omega/dk$ for this case. The formula holds in general.

3. [25 pts] Wave packets. Experiment with the Java applet on the webpage:
`http://phet.colorado.edu/simulations/sims.php?sim=Fourier_Making_Waves`

a) Sketch the series of waves formed by adding each of the coefficents in succession:
$A_1 = 1.27$, $A_3 = 0.52$, $A_5 = 0.25$, $A_7 = 0.18$, $A_9 = 0.14$, $A_{11} = 0.11$.
Why are all of the even terms 0?

b) Determine the coefficients needed to reproduce the following "inverted parabola" wave:



c) Solve a "Wave Game" puzzle of level 8 or higher, and attach the printed result (or email me a screendump).

d) In the "Discrete to Continuous" panel, explain what the three plots represent. Describe the variables k_1, λ_1, k_0, σ_k, and σ_x.

e) What is the effect of changing k_1 on the resulting amplitude distribution and wave packet? What happens as k_1 goes to zero?

f) Repeat for k_0 and σ_k.

g) Explain what Heisenberg uncertainty principle has to do with the frequency components of a wave packet.

Also: Tipler Chapter 5: #16, 25, 27, 37, 41.