**University of Kentucky, Physics 361**
**Problem Set #6, due Mon, Mar 29**

1. [15 pts] Different forms of the Laplacian. Show that the following operators (radial part of the Laplacian) are equal. Hint: put an arbitrary function $R(\rho)$ or $R(r)$ to the right of each term and apply the product rule for derivatives.

a) Cylindrical coordinates:

$$\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} \quad = \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} \quad = \quad \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho} + \frac{1}{4\rho^2}$$

b) Spherical coordinates:

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \quad = \quad \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} \quad = \quad \frac{1}{r}\frac{\partial^2}{\partial r^2}r$$

2. [30 pts] 2-d particle in an infinite circular well. Let $V(\rho) = 0$ for $\rho < a$ and $V(\rho) = \infty$ for $\rho > a$ (free particle confined to a disk).

a) Write Schrödinger's equation for this particle in 2-d cylindrical coordinates.

b) Do a separation of variables $\psi(\rho,\phi) = R(\rho)\Phi(\phi)$ to obtain radial and azimuthal equations.

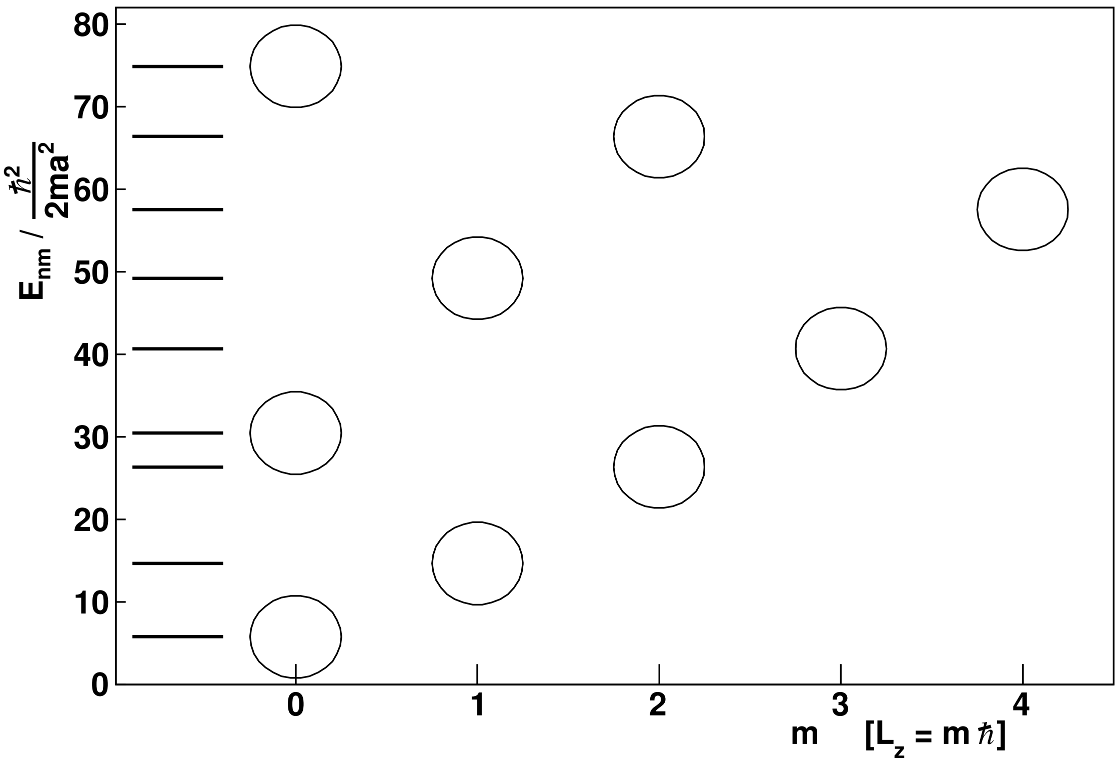
c) Show that $\Phi = e^{im\phi}$ is a solution of the azimuthal equation. Using boundary conditions at $\phi = 0 = 2\pi$, show that $m = 0, \pm 1, \pm 2, \ldots$.

d) Show that the radial equation is the Bessel equation of order $m$ with the substitution $x = k\rho$, where $E = \hbar^2 k^2/2m$. See `http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html` Equation (1). The solutions are called Bessel functions $J_m(k\rho)$ of order $m$.

e) Apply boundary condition $\psi(a) = 0$ to determine the values of $k$ for each $m$. For each value of $m$ there is a series of radial modes $k_{mn}$ indexed by $n$ (the number of nodes).

f) Show that the energy levels are $E_{nm} = \hbar^2 x_{nm}^2/2ma^2$, where $x_{nm}$ is the $n$-th zero of $J_m(x)$. Using the values of $x$ where the $J_m(x) = 0$ (see `http://mathworld.wolfram.com/BesselFunctionZeros.html`), fill in the numerical value of each energy level in the following chart. Draw the node lines for each mode.

g) For an electron, what radius $a$ would give a transition of $\lambda = 121.5$ nm from the first excited state to the ground state?




3. [20 pts] Show the following functions $Y_{lm}(\theta, \phi)$ are solutions of the differential equations:

$$\hat{L}_z Y_{lm} \equiv -i\hbar \frac{\partial}{\partial \phi} Y_{lm} = \hbar m Y_{lm} \quad \text{and}$$

$$\hat{L}^2 Y_{lm} \equiv \left( \frac{-\hbar^2}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \left( -\hbar^2 \frac{\partial^2}{\partial \phi^2} \right) \right) Y_{lm} = \hbar^2 l(l+1) Y_{lm}.$$

(In other words, they are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$).

a) $Y_{00} = 1$, b) $Y_{10} = \cos\theta$, c) $Y_{1\pm1} = \sin\theta e^{\pm i\phi}$, d) $Y_{2\pm1} = \sin\theta\cos\theta e^{\pm i\phi}$.

Show that the corresponding orbitals in rectangular coordinates are also solutions of $\nabla^2 r^l Y_{lm} = 0$ (where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$):

a) $Y_{00} = 1$ b) $rY_{10} = z$, c) $rY_{1\pm1} = x \pm iy$, d) $r^2 Y_{2\pm1} = z(x \pm iy)$.

Note: the real and imaginary parts can each be used as separate solutions.

Also: Tipler Chapter 7: #9, 10, 15, 16.