University of Kentucky, Physics 361
Problem Set #7, due Mon, Apr 05

1. [35 pts] Hydrogen radial wave functions: the hydrogen potential is $V(r) = -Zk_e e^2/r$.

a) Using the Laplacian ∇^2 in spherical coordinates, show that

$$\hat{T} = \frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\hat{L}^2}{2\mu r^2}$$

where the second term represents rotational kinetic energy, with

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right).$$

Write Schrödinger's equation for $\psi(r,\theta,\phi)$ of the hydrogen atom using this form of $\hat{T}$.

b) Make the substitution $\psi(r,\theta,\phi) = \frac{1}{r}u(r)Y_{lm}(\theta,\phi)$. The factor $\frac{1}{r}$ takes into account the spreading out of the wave function as it gets farther from the origin. Use the eigenvalue of Y_{lm},

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1)Y_{lm}$$

to simplify the equation.

c) Show that the result looks like the Schrödinger equation for $u(r)$ in one dimension with a centrifugal potential $V_c(r) = \hbar^2 l(l+1)/2\mu r^2$ in addition to the Coulomb potential. Compare with the potential of the centrifugal force $F = ma_c = mv^2/r$ using $L = mvr$.

d) Make the substitutions

$$\begin{aligned} u(r) &= U(\rho) \quad &\text{where} \quad &\rho = \frac{2r}{r_n}, \\ r_n &= \frac{na_0}{Z}, \quad &\text{where} \quad &a_0 = \frac{\hbar^2}{\mu k_e e^2}, \quad \text{and} \\ E_n &= \frac{-Z^2E_0}{n^2} \quad &\text{where} \quad &E_0 = \frac{\mu k_e^2 e^4}{2\hbar^2} \end{aligned}$$

to obtain the dimensionless equation

$$\left(\frac{\partial^2}{\partial\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{n}{\rho} - \frac{1}{4}\right)U(\rho) = 0.$$

e) Use repeated product rules to show that

$$(fgh)'' = f''gh + fg''h + fgh'' + 2f'g'h + 2f'gh' + 2fg'h'.$$

Use this and the substitution $U(\rho) = e^{-\rho/2}\rho^{l+1}L(\rho)$ to obtain Laguerre's differential equation

$$\rho L'' + (2l+2-\rho)L' + (n-l-1)L = 0$$

The solutions are associated Laguerre polynomials $L_{n-l-1}^{(2l+1)}(\rho)$.

f) Using the values of $L_\kappa^{(\alpha)}$ from http://mathworld.wolfram.com/LaguerrePolynomial.html, Eqs. 32–35, substitute back to find the radial wave functions $R_{nl}(r)$ for n =1, 2, and 3. Compare your answers with table $7 - 2$ in the text. What is the physical significance of κ?

g) Show that the ground-state wave function is normalized:

$$\int d^3r |\psi_{100}(r, \theta, \phi)|^2 = 1.$$

Also: Tipler Chapter 7: #26, 29, 39, 44, 73.