University of Kentucky, Physics 361
Problem Set #8, due Wed, Apr 14

1. [30 pts] While the spins of the two electrons in a hydrogen molecule must be antiparallel in the the ground state, there is a degeneracy due to the spin of the two protons, $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$.

a) Why doesn't the Pauli exclusion principle apply to the protons? In other words, why can the two protons have arbitrary m_s, while the electrons must occupy different states?

b) List the possible combinations of quantum numbers (m_{s_1}, m_{s_2}). What is the degeneracy of the ground state?

c) Show how the proton spins s_1 and s_2 couple to form a singlet and triplet. List the possible quantum numbers (s, m_s) of the total nuclear spin $\vec{s} = \vec{s}_1 + \vec{s}_2$ and find the degeneracy g_s for each value of total spin s. This explains the names 'singlet' and 'triplet'.

d) Ortho-hydrogen (triplet) is 15 meV higher in energy than the ground state of para-hydrogen $E = 0$ (singlet). For which value of s is it possible to have $(m_{s_1} = m_{s_2}) = \frac{1}{2}$ (the spins aligned)?

e) Using g_s and $f_{MB} = e^{-\epsilon/kT}$, calculate and plot the fraction of ortho- and para-hydrogen as a function of temperature from $T = 0 - 300$ K.

f) Calculate and plot the average energy per molecule as a function of temperature.

2. [30 pts] Helium atoms have total spin $s = 0$, and therefore the Bose-Einstein distribution

$$f_{BE}(\epsilon) = \frac{1}{e^{\alpha} e^{\epsilon/kT} - 1}.$$

a) Starting with $h\, dn_x = dp_x\, dx$ where dn_x is the number of available states in the range dp_x of momentum and dx in space in the x-direction, show that $dn = 4\pi p^2 dp V/h^3$ in three dimensions, where V is the volume and $dn = dn_x dn_y dn_z$.

b) Calculate the degeneracy $g(\epsilon)$ of states with kinetic energy $\epsilon = p^2/2m$, defined by $dn = g(\epsilon) d\epsilon$.

c) Insert $g(\epsilon)$ and $f(\epsilon)$ into $N = \int_0^\infty d\epsilon\, g(\epsilon)\, f_{BE}(\epsilon)$ to get a formula for the N, the total number of atoms. Show that the number density is

$$\frac{N}{V} = \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(\alpha) \qquad \text{where} \qquad I(\alpha) = \int_0^\infty \frac{\sqrt{x}dx}{e^{\alpha + x} - 1}.$$

d) Plot the integrand of $I(\alpha)$ for $\alpha = 0$, 0.5, and 1 to show that the integral decreases as a function of α. Circle the singularity in the integrand for $\alpha = 0$. Numerically, $I(0) = 2.315$, $I(0.5) = 0.7183$, and $I(1) = 0.3797$.

e) In the above formula, α changes as a function of temperature T to keep the number density N/V fixed at the experimental density of liquid helium, $\rho = 0.146$ g/mL at T_λ. Calculate the temperature corresponding to $\alpha = 0$, 0.5, and 1.

f) Because of the singularity at $\alpha = 0$, the value of α must remain positive, even as T drops below the critical temperature T_λ (when α=0). Therefore, the integral $I(\alpha) = I(0)$ is a constant for $T < T_\lambda$, and the density of the "normal" fluid component of helium drops at lower temperatures for $T < T_\lambda$. The rest of the atoms condense into the ground state to form a "superfluid" component with zero viscosity. Plot the density of the normal and superfluid components of Helium II as a function of T below the lambda point T_λ, keeping the total density constant. Note: the experimental value for the critical point is $T_\lambda = 2.17$ K.

3. [30 pts] A neutron star occurs when a star of mass up to 1.5 solar masses collapses under its own weight. Protons in the nuclei decay into neutrons, which have no electrical repulsion, and the only thing which prevents further collapse into a black hole (besides the repulsive nuclear force) is the Fermi repulsion of the neutrons, which have spin ($s = \frac{1}{2}$). This is a variant of the Pauli exclusion principle, which prevents the neutrons from all collapsing into the same state. They follow the Fermi-Dirac distribution

$$f_{FD}(\epsilon) = \frac{1}{e^{\alpha} e^{\epsilon/kT} + 1}.$$

a) Calculate the total number of neutrons created from a star 1.5 times the mass of the sun, $m_\odot = 1.9891 \times 10^{30}$ kg. (think big)

b) As in problem #2, show that the density of states is $dN = 8\pi p^2 dp V/h^3$. The extra factor of 2 comes from the two spin states of the neutron. In the zero temperature limit, each state is filled with one neutron, and all of the states are filled up to the maximum (Fermi) momentum p_F. Calculate the total number of neutrons $N = \int_0^{p_F} dN$ in a spherical volume (star) of radius R.

c) Calculate the weighted average kinetic energy of each neutron as a function of p_F, using

$$\langle E_{kin} \rangle = \frac{\int_0^{p_F} dN \frac{p^2}{2M_n}}{\int_0^{p_F} dN} = \frac{\int_0^{p_F} p^2 dp \frac{p^2}{2M_n}}{\int_0^{p_F} p^2 dp}.$$

d) Substitute p_F from part c) to show that the average kinetic energy of a neutron is

$$\langle E_{kin} \rangle = \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9\pi N}{4} \right)^{2/3}$$

e) The average potential energy from gravitational attraction is

$$\langle E_{pot} \rangle = -\frac{3}{5} \frac{G N M_n^2}{R}.$$

Minimize the total energy $\langle E_{tot} \rangle = \langle E_{kin} \rangle + \langle E_{pot} \rangle$ as a function of R to show that the equilibrium radius of the star is

$$R = \frac{\hbar^2 (9\pi/4)^{2/3}}{G M_n^3 N^{1/3}}.$$

Evaluate this expression numerically (the answer is close to the measured value of 12 km).

Also: Tipler Chapter 8: #11, 12, 23, and 26.