Problem 1

Comparison of Rayleigh-Jeans and Plank formulas for the black body spectrum.

Part a

Draw the node lines for the $(n_x, n_y) = (1, 1)$; $(1, 2)$; $(2, 1)$; and $(2, 2)$ standing wave modes on a square medium with sides of length L. Show that $n_x \lambda_x = 2L$. Show that the density of modes is $G(f) = 8\pi f^2/c^3$. It is defined as the number of modes per volume per frequency interval, i.e. $G(f)\, df \equiv d^3n/V$. Rememb er that $n_{x,y,z} > 0$ and there are two independent polarizations of light.

Figures 1 through 4 show the standing wave nodes for the given modes on a square medium with sides of length L.

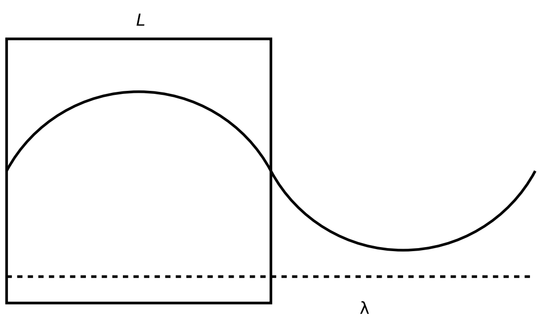

Figure 1: Standing Wave Modes: $(n_x, n_y) = (1, 1)$

Figures 1 and 3 show how a standing wave in the medium is distributed across the medium. In Figure 1, only half of the total wavelength of the wave is in the medium. In Figure 3, the entire wave is within the medium. Figure 1 shows that:

$$\begin{aligned} n_x \lambda_x &= 2L \\ (1)\, \lambda_x &= 2L \\ \frac{1}{2}\lambda_x &= L \end{aligned}$$

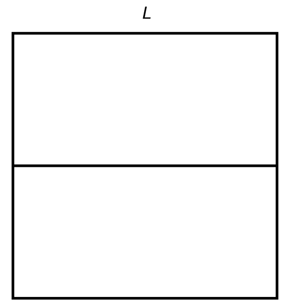

Figure 2: Standing Wave Modes: $(n_x, n_y) = (1, 2)$

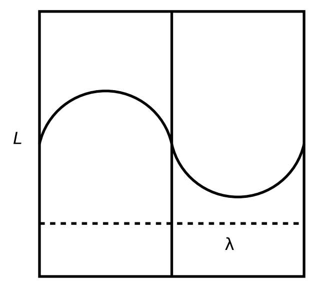

Figure 3: Standing Wave Modes: $(n_x, n_y) = (2, 1)$

Figure 3 shows that:

$$\begin{aligned} n_x \lambda_x &= 2L \\ 2\lambda_x &= 2L \\ \lambda_x &= L \end{aligned}$$

(Note, this next demonstration required [Tipler & Llewellyn, pp. 333-334] to complete.) To show that $G(f) = 8\pi f^2/c^3$, we start by recognizing that $G(f) = \frac{dN}{V\, df}$, where N is the number of states of the system inside of a given shell of radius n, V is volume and f is the frequency of the system. We continue by recognizing that the modes of the system, represented in n_x, n_y, and n_z, can only be positive integers. The state space, then, can be represented by $\frac{1}{8}$th of a sphere. the part of the sphere in the positive octant. So we get:

$$N = \left(\frac{1}{8}\right)\left(\frac{4\pi n^3}{3}\right)$$

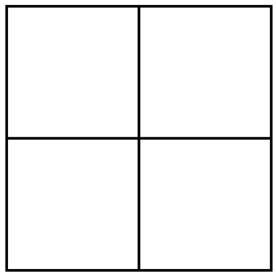

Figure 4: Standing Wave Modes: $(n_x, n_y) = (2, 2)$

$$= \frac{\pi n^3}{6}$$

And now, we can easily calculate dN in terms of dn:

$$dN = \frac{\pi 3n^2}{6}\,dn = \frac{\pi n^2}{2}\,dn$$

The infinitesimal frequency, df, can by found in terms of the infinitesimal number of modes in any one direction, dn, by:

$$\begin{aligned}
2L &= n\lambda \\
n\lambda &= \frac{nc}{f} \\
L &= \frac{nc}{2f} \\
n &= \frac{2fL}{c} \\
df &= \frac{c}{2L}\,dn
\end{aligned}$$

$G(f)$ is then:

$$\begin{aligned}
G(f) = \frac{dN}{V df} &= \frac{\frac{\pi n^2}{2}\,dn}{V\frac{c}{2L}\,dn} \\
G(f) &= \left(\frac{\pi n^2}{2}\right)\left(\frac{2L}{Vc}\right) \\
&= \frac{\pi n^2 L}{Vc}
\end{aligned}$$

Substituting in $n = \frac{2fL}{c}$:

$$\begin{aligned}
G(f) &= \frac{\pi\left(\frac{2fL}{c}\right)^2 L}{Vc} \\
&= \frac{4\pi f^2 L^3}{Vc^3}
\end{aligned}$$

But $L^3 = V$, and we actually have twice as many states since there are two polarities of light, so we end up with:

$$G(f) = \frac{8\pi f^2}{c^3}$$

Part b

The probability of a mode having energy ϵ is proportional to $e^{-\epsilon/kT}$, the Boltzman distribution. Let $\beta = 1/kT$, and integrate the total probability $Z = \int_0^\infty e^{-\beta\epsilon}d\epsilon$ to obtain the normalization factor. $Z(\beta)$ is also called the partition function. Show that $\langle\epsilon\rangle = -d\ln Z/d\beta = kT$. Show that this leads to the Rayleigh-Jeans formula for the spectral intensity of black body radiation.

The integral $Z = \int_0^\infty e^{-\beta\epsilon}d\epsilon$ is relatively straightforward to find:

$$\begin{aligned}
Z &= \int_0^\infty e^{-\beta\epsilon}d\epsilon \\
&= \left[-\frac{1}{\beta}e^{-\beta\epsilon}\right]_0^\infty \\
&= \left(-\frac{1}{\beta}\right)e^{-\beta(\infty)} - \left(-\frac{1}{\beta}\right)e^{-\beta(0)} \\
&= 0 + \frac{1}{\beta} \\
Z &= \frac{1}{\beta} \\
Z &= \frac{1}{\frac{1}{kT}} \\
Z &= kT
\end{aligned}$$

To find $\langle \epsilon \rangle$, we use the definition $\langle u \rangle = \frac{\int u f(x)\, dx}{\int f(x)\, dx}$, where, in this case, $f(x) = e^{-\beta\epsilon}$, and we integrate over the domain of ϵ, which is from 0 to ∞, since there are not negative energy states . We've already calculated $\int_0^\infty f(\epsilon)\, d\epsilon = kT$. This leaves $\int_0^\infty \epsilon e^{-\beta\epsilon} d\epsilon$. Note, however, that $\epsilon e^{-\beta\epsilon} = -\frac{d}{d\beta} e^{-\beta\epsilon}$, so we can write:

$$\begin{aligned}
\int_0^\infty \epsilon e^{-\beta\epsilon} d\epsilon &= \int_0^\infty -\frac{d}{d\beta} e^{-\beta\epsilon} d\epsilon \\
&= -\frac{d}{d\beta} \int_0^\infty e^{-\beta\epsilon} d\epsilon \\
&= -\frac{dZ}{d\beta} \\
&= \frac{1}{\beta^2} \\
&= kT^2
\end{aligned}$$

$\langle \epsilon \rangle$, then, becomes:

$$\langle \epsilon \rangle = \frac{(kT)^2}{kT} = kT$$

Also:

$$\begin{aligned}
-\frac{d\, \ln(\frac{1}{\beta})}{d\beta} &= -\frac{1}{\frac{1}{\beta}} - \beta^{-2} \\
&= \frac{\beta}{\beta^2} \\
&= \frac{1}{\beta} \\
&= kT
\end{aligned}$$

It's also possible to show this by noting that $d \ln Z = \frac{1}{Z} dZ$, so we have:

$$\begin{aligned}
\langle \epsilon \rangle &= -\frac{\frac{dZ}{d\beta}}{Z} \\
&= -\frac{1}{Z}\frac{dZ}{d\beta} \\
&= -\frac{d\,(\ln Z)}{d\beta}
\end{aligned}$$

Now that we have $G(f)$ and $\langle \epsilon \rangle$, we can calculate the Rayleigh-Jeans equation, $u(f) = G(f) \langle \epsilon \rangle$:

$$\begin{aligned}
u(f) &= G(f) \langle \epsilon \rangle \\
&= \frac{8\pi f^2}{c^3} kT
\end{aligned}$$

Part c

> Assume the energy is not continuous, but quantized to discrete levels $\epsilon = nhf$ for $n = 0, 1, 2, \ldots$ Repeat the calculation of the normalization factor $Z = \sum_{n=0}^\infty e^{-\beta\epsilon_n}$. Calculate $\langle \epsilon \rangle$. Show that this leads to Planck's formula.

The normalization factor, in this case, can be found by recognizing that $Z = \sum_{n=0}^\infty \left(e^{-\beta hf}\right)^n = \sum_{n=0}^\infty \left(e^{-hf/kT}\right)^n = \frac{1}{1-e^{-hf/kT}}$. As in part (b), we can note that:

$$\begin{aligned}
\epsilon_n e^{-\beta\epsilon_n} &= -\frac{d}{d\beta} e^{-\beta\epsilon_n} \\
&= -\frac{dZ}{d\beta} \\
&= -\frac{d}{d\beta}\frac{1}{1-e^{-\beta hf}} \\
&= -\frac{(-1)\left(-e^{-\beta hf}\right)(-hf)}{\left(1-e^{-\beta hf}\right)^2} \\
&= \frac{hfe^{-\beta hf}}{\left(1-e^{-\beta hf}\right)^2}
\end{aligned}$$

So once again:

$$\begin{aligned}
\langle \epsilon \rangle &= \frac{-\frac{dZ}{d\beta}}{Z} \\
&= \frac{\frac{hfe^{-\beta hf}}{(1-e^{-\beta hf})^2}}{\frac{1}{1-e^{-hf/kT}}} \\
&= \left(\frac{hfe^{-\beta hf}}{\left(1-e^{-\beta hf}\right)^2}\right)\left(1-e^{-hf/kT}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{hfe^{-\beta hf}}{1-e^{-\beta hf}} \\
&= \frac{hf}{\frac{1}{e^{-\beta hf}}-1} \\
&= \frac{hf}{e^{\beta hf}-1} \\
&= \frac{hf}{e^{hf/kT}-1}
\end{aligned}
$$

Now, to derive Planck's formula, we put $\langle\epsilon\rangle$ together with $G(f)$ to find $u(f)$:

$$
\begin{aligned}
u(f) &= G(f)\,\langle\epsilon\rangle \\
&= \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT}-1}
\end{aligned}
$$

Part d

> Show that $\langle\varepsilon\rangle = kT$ in the limit $hf \ll kT$. Show that $\langle\epsilon\rangle \approx 0$ in the limit $hf \gg kT$, thus solving the ultravilot catastrophe. Qualitatively, this is because the temperature is too low to excite even one photon (n=1).

If $hf \ll kT$, we can use the Taylor expansion for e^x:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Ignoring the x^2 and higher terms, we get:

$$e^{hf/kT} = 1 + \frac{hf}{kT}$$

And:

$$
\begin{aligned}
\langle\epsilon\rangle &= \frac{hf}{hf/kT+1-1} \\
&= \frac{hf}{hf/kT} \\
&= kT
\end{aligned}
$$

For $hf \gg kT$, we have:

$$\langle\epsilon\rangle = \frac{hf}{e^{hf}-1}$$

Since, by l'Hopital's rule, $e^{hf}-1 \to \infty$ faster than $hf \to \infty$,$\langle\epsilon\rangle \approx 0$.

Problem 2

> Some of the fundamental constants in SI units are:
>
> $$
> \begin{aligned}
> h &= 6.62607 \times 10^{-34}\,\text{J}\cdot\text{s} \\
> c &= 299792458\,\text{m/s} \\
> e &= 1.60218 \times 10^{-19}\text{C} \\
> k_e &= 8.98755 \times 10^{9}\text{N}\cdot\text{m}^2/\text{C}^2 = \frac{1}{4\pi\epsilon_0} \\
> k_B &= 1.38065 \times 10^{-23}\,\text{J/K} \\
> m_e &= 9.10938 \times 10^{-31}\text{kg} \\
> m_p &= 1.67262 \times 10^{-27}\text{kg} \\
> m_n &= 1.67493 \times 10^{-27}\text{kg}
> \end{aligned}
> $$
>
> Note that $1\text{eV} = e \cdot 1\text{V}$ is a compound unit of energy. Calculate the following useful combinations of constants in the units specified: hc [eV · nm], $\hbar c = \frac{h}{2\pi}c$ [MeV · fm], $k_e e^2$ [eV · nm], $\alpha = k_e e^2/\hbar c$ [1], kT [meV] at room temperature $T = 20°\text{C} = 293.15\,\text{K}$, and m_e, m_p, m_n $\left[\text{MeV}/c^2\right]$. We will use these combinations in natural units repeatedly throughout the rest of the semester.

NOTE: The mass of the proton and neutron in kg as stated in the original problem is incorrect. These should be $\times 10^{-27}$, not $\times 10^{-24}$.

hc [eV · nm]:

$$
\begin{aligned}
hc &= (6.62607 \times 10^{-34}\,\text{J}\cdot\text{s})(299792458\,\text{m/s}) \\
&= 1.98644 \times 10^{-25}\,\text{J}\cdot\text{m}
\end{aligned}
$$

We convert this by dividing througy by e and multiplying by 10^9nm/m:

$$\begin{aligned} hc &= \left(\frac{1.98644\times 10^{-25}\,\mathsf{J\cdot m}}{1.60218\times 10^{-19}\mathsf{C}}\right)\left(10^9\mathsf{nm/m}\right) \\ &= 1239.84\,\mathsf{eV\cdot nm} \end{aligned}$$

$\hbar c = \frac{h}{2\pi}c$ [MeV · fm]:

$$\begin{aligned} \hbar c &= \left(\frac{1239.84\,\mathsf{eV\cdot nm}}{2\pi}\right)\left(\frac{\mathsf{MeV}}{10^6\mathsf{eV}}\right)\left(\frac{10^6\mathsf{fm}}{\mathsf{nm}}\right) \\ &= 197.327\,\mathsf{MeV\cdot fm} \end{aligned}$$

$k_e e^2$ [eV · nm]:

$$\begin{aligned} k_e e^2 &= \left(8.98755\times 10^9\frac{\mathsf{N\cdot m^2}}{\mathsf{C^2}}\right)\left(1.60218\times 10^{-19}\mathsf{C}\right) \\ &= 1.43997\ \times 10^{-9}\mathsf{eV\cdot m} \end{aligned}$$

We convert this by multiplying by 10^9nm/m:

$$\begin{aligned} k_e e^2 &= \left(1.43997\times 10^{-9}\mathsf{eV\cdot m}\right)\left(10^9\mathsf{nm/m}\right) \\ &= 1.43997\,\mathsf{eV\cdot nm} \end{aligned}$$

$\alpha = k_e e^2/\hbar c$ [1]:

$$\begin{aligned} \alpha &= \frac{k_e e^2}{\hbar c} \\ &= \frac{1.43997\,\mathsf{eV\cdot nm}}{197.327\,\mathsf{eV\cdot nm}} \\ &= 7.29735\times 10^{-3} \\ &= \frac{1}{137.05} \end{aligned}$$

kT [meV] at room temperature $T = 20°\mathsf{C} = 293.15\,\mathsf{K}$:

$$\begin{aligned} kT &= \frac{1.38065\times 10^{-23}\mathsf{J/K}}{1.60218\times 10^{-19}\mathsf{C}}(293.15\,\mathsf{K}) \\ &= \left(2.52617\times 10^{-2}\mathsf{eV}\right)\left(\frac{1000\,\mathsf{meV}}{\mathsf{eV}}\right) \\ &= 25.2617\,\mathsf{meV} \\ &= \frac{1}{40}\,\mathsf{eV} \end{aligned}$$

m_e, m_p, m_n $\left[\mathsf{MeV}/c^2\right]$:

For each, we find mc^2, dividing the mass by the elementary charge and multiplying by c^2 to find the energy.

$$\begin{aligned} m_e c^2 &= \left(\frac{9.10938\times 10^{-31}\mathsf{kg}}{1.60218\times 10^{-19}\mathsf{C}}\right)(299792458\,\mathsf{m/s})^2 \\ m_e &= 0.510998\,\mathsf{MeV}/c^2 \end{aligned}$$

$$\begin{aligned} m_p c^2 &= \left(\frac{1.67262\times 10^{-27}\mathsf{kg}}{1.60218\times 10^{-19}\mathsf{C}}\right)(299792458\,\mathsf{m/s})^2 \\ m_p &= 938.271\,\mathsf{MeV}/c^2 \end{aligned}$$

$$\begin{aligned} m_n c^2 &= \left(\frac{1.67493\times 10^{-27}\mathsf{kg}}{1.60218\times 10^{-19}\mathsf{C}}\right)(299792458\,\mathsf{m/s})^2 \\ m_n &= 939.565\,\mathsf{MeV}/c^2 \end{aligned}$$

Problem 3.24

> The wavelengths of visible light range from about 380 nm to about 750 nm.

Part a

> What is the range of photon energies (in eV) in visible light?

Each photon has energy $E = hf = hc/\lambda$. $hc \approx 1240\,\mathsf{eV\cdot nm}$. So at 380 nm, each photon has energy:

$$E \approx \frac{1240\,\mathsf{eV\cdot nm}}{380\,\mathsf{nm}} \approx 3.26\,\mathsf{eV}$$

At 750 nm, each photon has energy:

$$E \approx \frac{1240\,\mathsf{eV\cdot nm}}{750\,\mathsf{nm}} \approx 1.65\,\mathsf{eV}$$

Part b

> A typical FM radio station's broadcast frequency is about 100 MHz. What is the energy of an FM photon of the frequency?

NOTE: I assume they meant "of **that** frequency".

Each photon has energy $E = hf$. So, at 100 MHz, each photon has energy:

$$\begin{aligned} E &= \left(\frac{6.62607\times10^{-34}\,\mathsf{J\cdot s}}{1.60218\times10^{-19}\,\mathsf{C}}\right)(100\,\mathsf{MHz}) \\ &\approx 4.14\times10^{-7}\mathsf{eV} \end{aligned}$$

Problem 3.26

> The work function for cesium is 1.9 eV, the lowest of any metal.

Part a

> Find the threshold frequency and wavelength for the photoelectric effect.

The work function is $\phi = hf_t = \frac{hc}{\lambda_t}$. Since h and hc are just constants, f_t and λ_t can be easily found:

$$\begin{aligned} f_t &= \frac{\phi}{h} \\ &= \frac{(1.9\,\mathsf{V})\left(1.60218\times10^{-19}\mathsf{C}\right)}{6.62607\times10^{-34}\,\mathsf{J\cdot s}} \\ &\approx 4.59\times10^{14}\mathsf{Hz} \end{aligned}$$

$$\begin{aligned} \lambda_t &= \frac{hc}{\phi} \\ &\approx \frac{1240\,\mathsf{eV\cdot nm}}{1.9\,\mathsf{eV}} \\ &\approx 653\,\mathrm{nm} \end{aligned}$$

Part b

> Find the stopping potential if the wavelength of the incident light is 300 nm.

The stopping potential is given by $eV_0 = hf - \phi$, or $V_0 = \frac{hf-\phi}{e}$. However, we're given the wavelength, not the frequency, so we must use this form of the same equation:

$$\begin{aligned} V_0 &= \frac{\frac{hc}{\lambda} - \phi}{e} \\ &\approx \frac{\frac{1240\,\mathsf{eV\cdot nm}}{300\,\mathsf{nm}} - 1.9\,\mathsf{eV}}{e} \\ &\approx 2.23\,\mathrm{V} \end{aligned}$$

(Note that the $1/e$ cancels with the e in eV to leave volts.

Part c

> Find the stopping potential if the wavelength of the incident light is 400 nm.

Similar to part b:

$$\begin{aligned} V_0 &= \frac{\frac{hc}{\lambda} - \phi}{e} \\ &\approx \frac{\frac{1240\,\mathsf{eV\cdot nm}}{400\,\mathsf{nm}} - 1.9\,\mathsf{eV}}{e} \\ &\approx 1.20\,\mathrm{V} \end{aligned}$$

Problem 3.31

> Under optimum conditions, the eye will perceive a flash if about 60 photons arrive at the cornea. How much energy is this in joules if the wavelength of the light is 550 nm?

The energy of each photon $E = \frac{hc}{\lambda}$:

$$\begin{aligned} E &= \frac{hc}{\lambda} \\ &\approx \frac{1240\,\text{eV}\cdot\text{nm}}{550\,\text{nm}} \\ &\approx 2.25\,\text{eV} \end{aligned}$$

If 60 photons with this energy arrive at the cornea, this becomes $(60\,\text{photons})\,(2.25\,\text{eV/photon}) = 135\,\text{eV}$. To convert this to Joules, just multiply by e:

$$\begin{aligned} E &\approx (135\,\text{eV})\left(1.60218\times10^{-19}\text{C}\right) \\ &\approx 2.17\times10^{-17}\,\text{J} \end{aligned}$$

Problem 3.40

> Compton's equation (Equation 3-25) indicates that a graph of λ_2 versus $(1-\cos\theta)$ should be a straight line whose slope h/mc allows a determination of h. Given that the wavelength of λ_1 in Figure 3-17 is 0.0711 nm, compute λ_2 for each scattering angle in the figure and graph the result versus $(1-\cos\theta)$. What is the slope of the line?

Using the calibration of the Bragg spectrometer, $\theta_s = 6°42' = 6.7°$, and $2d\sin\theta = m\lambda$, we can determine $\frac{2d}{m} = \frac{0.0711\,\text{nm}}{\sin 6.7°} = 0.6094\,\text{nm}$. From that, we can calculate λ_2 values using $\lambda_2(\theta_B) = \left(\frac{2d}{m}\right)\sin\theta_B$, taking θ_B from the graphs in [Tipler & Llewellyn, p.137]. We can also calculate $1-\cos\theta_c$ using the angles specified for each graph.

This yields the following for the function $\lambda_2(\theta_B)$:

$$\begin{aligned} \lambda_2(6°42') &= 0.0711\,\text{nm} \\ \lambda_2(6°47') &= 0.0720\,\text{nm} \\ \lambda_2(6°56') &= 0.0736\,\text{nm} \\ \lambda_2(7°06') &= 0.0753\,\text{nm} \end{aligned}$$

The corresponding values for $f(\theta_C) = 1-\cos\theta_C$ are:

$$\begin{aligned} f(0) &= 0 \\ f(45°) &= 0.293 \\ f(90°) &= 1 \\ f(135°) &= 1.707 \end{aligned}$$

Plotting λ_2 versus f, we get Figure 5.

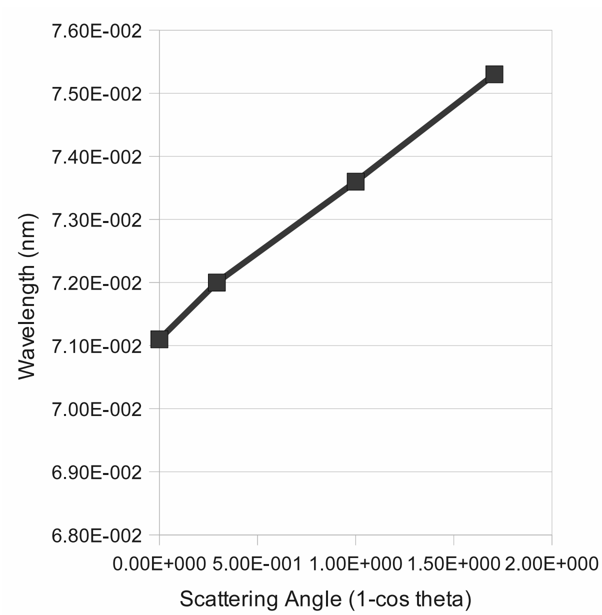

Figure 5: Compton Scattering for $\lambda_1 = 0.0711\,\text{nm}$

The y-intercept of the best fit line for that graph is 0.07117 nm, and the slope is $0.002432\,\text{nm} = \lambda_c$. The $\chi^2 = 1.59\times10^{-8}$nm, and $\delta y = \sqrt{\frac{\chi^2}{2}} \approx 0.0009\,\text{nm}$.

Problem 3.47

> When a beam of monochromatic x-rays is incident on a particular NaCl crystal, Bragg reflection in the first order (i.e., with $m = 1$) occurs at $\theta = 20°$. The value of $d = 0.28\,\text{nm}$. What is the minimum voltage at which the x-ray tube can be operating?

Bragg's Law states that $2d\sin\theta = m\lambda$, and we also know that $\lambda_m = \frac{1240\,\text{eV}\cdot\text{nm}}{V_m}$. Since $m = 1$,we can easily solve for V_m:

$$\begin{aligned} eV_m &= \frac{1240\,\text{eV}\cdot\text{nm}}{2d\sin\theta} \\ &= \frac{1240\,\text{eV}\cdot\text{nm}}{2(0.28\,\text{nm})\sin 20^\circ} \\ &= 6474\,\text{eV} \end{aligned}$$

Dividing through by e to get volts yields $6474\,\text{V}$.

Problem 3.49

> Show that the maximum kinetic energy, E_k, called the Compton edge, that a recoiling electron can carry away from a Compton scattering event is given by:
>
> $$E_k = \frac{hf}{1 + mc^2/2hf} = \frac{2E_\gamma^2}{2E_\gamma + mc^2}$$

[Wikipedia helped with this one slightly.] We know that the relationship between the wavelengths of the light and the angle of scattering is given by the equation:

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta)$$

We also know that wavelength is related to energy by the equation:

$$E_\gamma = hf = \frac{hc}{\lambda}$$

Substituting into the first equation, we get:

$$\begin{aligned} \frac{hc}{E_2} - \frac{hc}{E_1} &= \frac{h}{mc}(1 - \cos\theta) \\ \frac{1}{E_2} - \frac{1}{E_1} &= \frac{1}{mc^2}(1 - \cos\theta) \end{aligned}$$

$$\begin{aligned} \frac{1}{E_2} &= \frac{(1 - \cos\theta)}{mc^2} + \frac{1}{E_1} \\ \frac{E_1}{E_2} &= \frac{E_1(1 - \cos\theta)}{mc^2} + 1 \\ E_2 &= \frac{E_1}{\frac{E_1(1-\cos\theta)}{mc^2} + 1} \end{aligned}$$

$E_1 - E_2 = E_k$ gives the energy transfered to the electron. To maximize E_k, we need to minimize E_2. To minimize E_2, we want $E_1(1 - \cos\theta)$ as large as possible, which will happen when $\cos\theta = -1$, or when $\theta = 180^\circ$. This, along with the substitution for E_2, (and using $E_\gamma = E_1$) gives:

$$\begin{aligned} E_\gamma - E_k &= \frac{E_\gamma}{\frac{2E_\gamma}{mc^2} + 1} \\ &= \frac{E_\gamma}{\frac{2E_\gamma + mc^2}{mc^2}} \\ &= \frac{E_\gamma mc^2}{2E_\gamma + mc^2} \\ E_k &= E_\gamma - \frac{E_\gamma mc^2}{2E_\gamma + mc^2} \\ &= \frac{E_\gamma\left(2E_\gamma + mc^2\right) - E_\gamma mc^2}{2E_\gamma + mc^2} \\ &= \frac{2E_\gamma^2 + E_\gamma mc^2 - E_\gamma mc^2}{2E_\gamma + mc^2} \\ E_k &= \frac{2E_\gamma^2}{2E_\gamma + mc^2} \end{aligned}$$

Now that we have this, we can easily substitue $E_\gamma = hf$ back into the equation to get the other form from the problem statement:

$$\begin{aligned} E_k &= \frac{2(hf)^2}{2hf + mc^2} \\ &= \frac{\frac{2(hf)^2}{2hf}}{1 + \frac{mc^2}{2hf}} \\ E_k &= \frac{hf}{1 + \frac{mc^2}{2hf}} \end{aligned}$$