Problem 1

Complex numbers, circular, and hyperbolic functions.

Part a

Use the power series of e^x, $\sin(\theta)$, $\cos(\theta)$ to prove Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.

The power series of e^x, $\sin(\theta)$, $\cos(\theta)$ are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\end{aligned}$$

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\end{aligned}$$

We can use these to get

$$\begin{aligned}\cos\theta + i\sin\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\ &\quad + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)\end{aligned}$$

We can also use these to get

$$\begin{aligned}e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{(\theta)^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} \\ &\quad - i\frac{\theta^7}{7!} + \frac{\theta^8}{8!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \\ &\quad + i\theta - i\frac{(\theta)^3}{3!} + i\frac{\theta^5}{5!} - i\frac{\theta^7}{7!} \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \\ &\quad + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\right) \\ &= \cos\theta + i\sin\theta\end{aligned}$$

Part b

Use Euler's identity to show if $z = x + iy$ where (x, y) are cartesian coordinates in the complex plane, then $z = re^{i\theta}$ where (r, θ) are the polar coordinates of the same point.

The most direct solution to this problem is to recognize that on the complex plane $x = r\cos\theta$ and $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, so:

$$\begin{aligned}z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta)\end{aligned}$$

But we know, from Euler's identity, that $e^{i\theta} = \cos\theta + i\sin\theta$, so:

$$z = re^{i\theta}$$

We can also work backwards. In polar coordinates, r represents the distance from the origin, and θ represents the angle made with the x axis. Given the cartesian coordinates (x, y), we have $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, or $\theta = \tan^{-1}\frac{y}{x}$. Euler's identity is $e^{i\theta} = \cos\theta + i\sin\theta$. So we have:

$$\begin{aligned}x + iy &= re^{i\theta} \\ &= \sqrt{x^2 + y^2}\left(e^{i\tan^{-1}\frac{y}{x}}\right) \\ &= \sqrt{x^2 + y^2}\left(\cos\left(\tan^{-1}\frac{y}{x}\right) + i\sin\left(\tan^{-1}\frac{y}{x}\right)\right)\end{aligned}$$

But, we know that, if $\theta = \tan^{-1}\frac{y}{x}$, then $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$. Similarly, we know that $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. So we get:

$$\begin{aligned} x+iy &= \sqrt{x^2+y^2}\left(\cos\left(\tan^{-1}\frac{y}{x}\right)+i\sin\left(\tan^{-1}\frac{y}{x}\right)\right) \\ &= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}+i\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right) \\ &= x+iy \end{aligned}$$

Part c

> The complex conjugate z^* is formed by replacing i with $-i$ everywhere in z. The modulus $|z| \equiv \sqrt{z^*z}$ is the complex analog of absolute value. Use $z = x+iy = re^{i\theta}$ to show the relations $|z|^2 = z^*z = zz^* = x^2+y^2 = r^2$. Thus the modulus is the distance of z from the origin.

Using $z = re^{i\theta}$ and the relationship described above between z^* and z, we have:

$$\begin{aligned} |z|^2 &= z^*z \\ &= re^{-i\theta}re^{i\theta} \\ &= re^{-i\theta}re^{i\theta} \\ &= r^2e^{-i\theta+i\theta} \\ &= r^2e^0 \\ &= r^2 \end{aligned}$$

Using $z = x+iy$ we have:

$$\begin{aligned} |z|^2 &= z^*z \\ &= (x-iy)(x+iy) \\ &= x^2+iyx-iyx-(iy)^2 \\ &= x^2-i^2y^2 \\ &= x^2+y^2 \end{aligned}$$

Finally, we have, because multiplication is commutative in $\mathbb{C}$ ($\mathbb{C}$ is a field):

$$\begin{aligned} z^*z &= (x-iy)(x+iy) \\ &= (x+iy)(x-iy) \\ &= zz^* \end{aligned}$$

So we have $|z|^2 = z^*z = zz^* = x^2+y^2 = r^2$.

Part d

> Expand z^2 in terms of x, y and also r, θ to see why $|z|^2$ is more useful in general than z^2.

Expanding z^2 in terms of x and y, we get:

$$\begin{aligned} z^2 &= (x+iy)^2 \\ &= x^2+2xiy+(iy)^2 \\ &= x^2+2ixy-y^2 \end{aligned}$$

Expanding z^2 in terms of r and θ, we get:

$$\begin{aligned} z^2 &= \left(re^{i\theta}\right)^2 \\ &= r^2e^{2i\theta} \end{aligned}$$

Part e

> Multiply $e^{i\theta}$ by its complex conjugate and expand using Euler's identity to prove the relation $\sin^2\theta+\cos^2\theta = 1$. This shows that $e^{i\theta}$traces out a circle in the complex plane.

The complex conjugate of $e^{i\theta}$ is $e^{-i\theta}$, and Euler's identity is $e^{i\theta} = \cos\theta + i\sin\theta$. Using these, we get:but

$$\begin{aligned} \left|e^{i\theta}\right| &= e^{-i\theta}e^{i\theta} \\ &= e^{-i\theta}e^{i\theta} \\ &= (\cos\theta - i\sin\theta)(\cos\theta+i\sin\theta) \\ &= \cos^2\theta + i\sin\theta\cos\theta - i\sin\theta\cos\theta - (i\sin\theta)^2 \\ &= \cos^2\theta - i^2\sin^2\theta \\ &= \cos^2\theta+\sin^2\theta \end{aligned}$$

But, $e^{-i\theta}e^{i\theta} = e^{-i\theta+i\theta} = e^0 = 1$, so $\cos^2\theta+\sin^2\theta = 1$.

Part f

> Use Euler's identity on $e^{i\theta}$ and $e^{-i\theta}$ to express $\cos\theta$, $\sin\theta$ and $\tan\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$.

Euler's identity is $e^{i\theta} = \cos\theta + i\sin\theta$. From this, we know that $e^{-i\theta} = \cos\theta - i\sin\theta$. We can add, giving us

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ &= 2\cos\theta \end{aligned}$$

So we have that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$. We can either substitute this into Euler's identity to get:

$$\begin{aligned} e^{i\theta} &= \frac{e^{i\theta}+e^{-i\theta}}{2} + i\sin\theta \\ e^{i\theta} - \frac{e^{i\theta}+e^{-i\theta}}{2} &= i\sin\theta \\ \frac{2e^{i\theta} - e^{i\theta} - e^{-i\theta}}{2} &= i\sin\theta \\ \frac{e^{i\theta} - e^{-i\theta}}{2} &= i\sin\theta \\ \frac{e^{i\theta} - e^{-i\theta}}{2i} &= \sin\theta \end{aligned}$$

Or, we can find $\sin\theta$ like we found $\cos\theta$, but subtracting instead of adding:

$$\begin{aligned} e^{i\theta} - e^{-i\theta} &= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta \\ &= 2i\sin\theta \end{aligned}$$

This gives $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have:

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{\frac{e^{i\theta}-e^{-i\theta}}{2i}}{\frac{e^{i\theta}+e^{-i\theta}}{2}} \\ &= \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)} \\ &= i\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} \end{aligned}$$

Part g

> Using the similar definitions $\cosh(\alpha) \equiv \frac{1}{2}\left(e^{\alpha} + e^{-\alpha}\right)$ and $\sinh(\alpha) \equiv \frac{1}{2}\left(e^{\alpha} - e^{-\alpha}\right)$, derive the analog of Euler's identity for hyperbolic functions. Hint: i becomes $\pm$.

We can subtract $\sinh(\alpha)$ from $\cosh(\alpha)$ to get:

$$\begin{aligned} \cosh(\alpha) - \sinh(\alpha) &= \frac{1}{2}\left(e^{\alpha} + e^{-\alpha}\right) \\ &\quad -\frac{1}{2}\left(e^{\alpha} - e^{-\alpha}\right) \\ &= \frac{1}{2}\left(e^{\alpha} + e^{-\alpha} - e^{\alpha} + e^{-\alpha}\right) \\ &= \frac{1}{2}\left(e^{-\alpha} + e^{-\alpha}\right) \\ &= e^{-\alpha} \end{aligned}$$

If we use $-\alpha$ instead of α, we get:

$$\begin{aligned} \cosh(-\alpha) - \sinh(-\alpha) &= \frac{1}{2}\left(e^{-\alpha} + e^{\alpha}\right) \\ &\quad -\frac{1}{2}\left(e^{-\alpha} - e^{\alpha}\right) \\ \cosh(\alpha) + \sinh(\alpha) &= \frac{1}{2}\left(e^{-\alpha} + e^{\alpha} - e^{-\alpha} + e^{\alpha}\right) \\ &= \frac{1}{2}\left(e^{\alpha} + e^{\alpha}\right) \\ &= e^{\alpha} \end{aligned}$$

So we have $\cosh(\alpha) \pm \sinh(\alpha) = e^{\pm\alpha}$.

Part h

> Multiply and expand e^{α} and $e^{-\alpha}$ in two ways to derive a similar formula as in part (e) for $\cosh(\alpha)$ and $\sinh(\alpha)$. This shows that $(\cosh(h), \sinh(\alpha))$ traces out a hyperbola in that plane.

We have $e^{\alpha}e^{-\alpha} = e^{\alpha-\alpha} = e^{0} = 1$. But, we also have:

$$\begin{aligned} e^{\alpha}e^{-\alpha} &= \left(\cosh(\alpha) + \sinh(\alpha)\right)\left(\cosh(\alpha) - \sinh(\alpha)\right) \\ &= \cosh^2(\alpha) - \sinh(\alpha)\cosh(\alpha) \\ &\quad + \sinh(\alpha)\cosh(\alpha) - \sinh^2(\alpha) \\ &= \cosh^2(\alpha) - \sinh^2(\alpha) \end{aligned}$$

So, we have $\cosh^2(\alpha) - \sinh^2(\alpha) = 1$

Part i

> Derive the addition formulas for $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$ by multiplying and expanding $e^{i\alpha} \cdot e^{\pm i\beta}$ and then separating the real and imaginary parts.

Using Euler's formula:

$$\begin{aligned} e^{i\alpha}e^{\pm i\beta} &= e^{i\alpha \pm i\beta} \\ &= e^{i(\alpha \pm \beta))} \\ &= \cos(\alpha \pm \beta) + i\sin(\alpha \pm \beta) \end{aligned}$$

But also, still using Euler's formula:

$$\begin{aligned} e^{i\alpha}e^{\pm i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin(\pm\beta)) \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta \pm i\sin\beta) \\ &= \cos\alpha\cos\beta \pm i\sin\beta\cos\alpha \\ &\quad + i\sin\alpha\cos\beta \pm i^2\sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta \pm i\sin\beta\cos\alpha \\ &\quad + i\sin\alpha\cos\beta \pm \sin\alpha\sin\beta \\ &= \cos\alpha\cos\beta \pm \sin\alpha\sin\beta \\ &\quad + i(\sin\alpha\cos\beta \pm \sin\beta\cos\alpha) \end{aligned}$$

Since the real parts correspond, and the imaginary parts correspond, we end up with

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$$

and

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$$

Part j

> Obtain the derivatives of $\sin\theta$ and $\cos\theta$, $\sinh\alpha$, and $\cosh\alpha$ using the derivative of $e^{i\theta}$ and $e^{\pm\alpha}$.

We can take the derivative of $e^{i\theta}$ directly:

$$\begin{aligned} \frac{d}{d\theta}e^{i\theta} &= ie^{i\theta} \\ &= i\cos\theta + i^2\sin\theta \\ &= i\cos\theta - \sin\theta \end{aligned}$$

Now, we can use the derivative of Euler's identity:

$$\begin{aligned} \frac{d}{d\theta}e^{i\theta} &= \frac{d}{dx}\cos\theta + \frac{d}{dx}i\sin\theta \\ &= \frac{d}{dx}\cos\theta + i\frac{d}{dx}\sin\theta \end{aligned}$$

So we have $-\sin\theta + i\cos\theta = \frac{d}{dx}\cos\theta + i\frac{d}{dx}\sin\theta$. Since the real parts correspond, and the imaginary parts correspond, we have $\frac{d}{dx}\cos\theta = -\sin\theta$ and $\frac{d}{dx}\sin\theta = \cos\theta$.

Unfortunately, this approach won't work for $\sinh(\alpha)$ and $\cosh(\alpha)$. For these, we can determine the derivatives by using $\sinh(\alpha) = \frac{1}{2}(e^{\alpha} - e^{-\alpha})$ and $\cosh(\alpha) = \frac{1}{2}(e^{\alpha} + e^{-\alpha})$. Using these, we can determine that:

$$\begin{aligned} \frac{d}{d\alpha}\sinh(\alpha) &= \frac{d}{d\alpha}\left[\frac{1}{2}\left(e^{\alpha} - e^{-\alpha}\right)\right] \\ &= \frac{1}{2}\frac{d}{d\alpha}\left[e^{\alpha} - e^{-\alpha}\right] \\ &= \frac{1}{2}\left[e^{\alpha} - (-1)e^{-\alpha}\right] \\ &= \frac{1}{2}\left[e^{\alpha} + e^{-\alpha}\right] \end{aligned}$$

But, this is just $\cosh(\alpha)$. So $\frac{d}{dx}\sinh(\alpha) = \cosh(\alpha)$. Similary, we have:

$$\begin{aligned} \frac{d}{d\alpha}\cosh(\alpha) &= \frac{d}{d\alpha}\left[\frac{1}{2}\left(e^{\alpha} + e^{-\alpha}\right)\right] \\ &= \frac{1}{2}\frac{d}{d\alpha}\left[e^{\alpha} + e^{-\alpha}\right] \\ &= \frac{1}{2}\left[e^{\alpha} + (-1)e^{-\alpha}\right] \\ &= \frac{1}{2}\left[e^{\alpha} - e^{-\alpha}\right] \end{aligned}$$

And this is just $\sinh(\alpha)$. So, $\frac{d}{dx}\cosh(\alpha) = \sinh(\alpha)$

Problem 2

Beats and group velocity.

Part a

> Show that two waves of equal frequency and amplitude travelling in opposite directions $Ae^{ikx-i\omega t}$ and $Ae^{-ikx-i\omega t}$ superimpose to form a standing wave. What is the resulting wavelength?

Superimposing waves can be added to find the composite wave function. This gives us:

$$\begin{aligned} Ae^{ikx-i\omega t} + Ae^{-ikx-i\omega t} &= Ae^{-i\omega t}\left(e^{ikx} + e^{-ikx}\right) \\ &= 2\cos(kx)\,Ae^{-i\omega t} \end{aligned}$$

The $\cos(kx)$ describes the standing wave in space, and the $e^{-i\omega t}$ describes the wave oscillations in time. The wavelength in space in $\lambda = \frac{2\pi}{k}$.

Part b

> Using exponentials, show that two pure waves $e^{i(kx-\omega t)}$ of frequency (ω_1, k_1) and (ω_2, k_2), superimpose to form a beat pattern of $2\cos(\Delta kx - \Delta\omega t)\,e^{i\bar{k}x-i\bar{\omega}t}$. Derive the formulas for the combined frequencies $(\Delta\omega, \Delta k)$ and $\left(\bar{\omega}, \bar{k}\right)$. Identify the wave packet and the phase (carrier) wave, and solve the velocity of each. What beat frequency do you hear?

These waves have equations $e^{i(k_1x-\omega_1t)}$ and $e^{i(k_2x-\omega_2t)}$. Let's define $\theta_1 = k_1x - \omega_1t$ and $\theta_2 = k_2x - \omega_2t$. If we define the average $\bar{\theta} = \frac{1}{2}(\theta_2 + \theta_1)$ and differential $\Delta\theta = \frac{1}{2}(\theta_2 - \theta_1)$, then we can define $\theta_1 = \bar{\theta} + \Delta\theta$ and $\theta_2 = \bar{\theta} - \Delta\theta$. Using these definitions and adding the wave functions, we get:

$$\begin{aligned} e^{i\theta_1} + e^{i\theta_2} &= e^{i\left(\bar{\theta}+\Delta\theta\right)} + e^{i\left(\bar{\theta}-\Delta\theta\right)} \\ &= e^{i\bar{\theta}+i\Delta\theta} + e^{i\bar{\theta}-i\Delta\theta} \\ &= e^{i\bar{\theta}}e^{i\Delta\theta} + e^{i\bar{\theta}}e^{-i\Delta\theta} \\ &= e^{i\bar{\theta}}\left(e^{i\Delta\theta} + e^{-i\Delta\theta}\right) \\ &= 2\cos(\Delta\theta)\,e^{i\bar{\theta}} \end{aligned}$$

To get back to an expression in terms of angular frequencies and spacial wave densities, we need to define the average $\bar{k} = \frac{1}{2}(k_2 + k_1)$ and $\bar{\omega} = \frac{1}{2}(\omega_2 + \omega_1)$, and we need to define the differentials $\Delta k = \frac{1}{2}(k_2 - k_1)$ and $\Delta\omega = \frac{1}{2}(\omega_2 - \omega_1)$. Using the definitions $\theta_1 = k_1x - \omega_1t$, $\theta_2 = k_2x - \omega_2t$, $\bar{\theta} = \frac{1}{2}(\theta_2 + \theta_1)$ and $\Delta\theta = \frac{1}{2}(\theta_2 - \theta_1)$ we can get:

$$\begin{aligned} \bar{\theta} &= \frac{1}{2}\left((k_2x - \omega_2t) + (k_1x - \omega_1t)\right) \\ &= \frac{1}{2}(k_2x + k_1x - \omega_2t - \omega_1t) \\ &= \frac{1}{2}\left((k_2 + k_1)x - (\omega_2 + \omega_1)t\right) \\ &= \frac{1}{2}(k_2 + k_1)x - \frac{1}{2}(\omega_2 + \omega_1)t \end{aligned}$$

Now we can substitute $\bar{k} = \frac{1}{2}(k_2 + k_1)$ and $\bar{\omega} = \frac{1}{2}(\omega_2 + \omega_1)$, so we get:

$$\bar{\theta} = \bar{k}x - \bar{\omega}t$$

Similarly:

$$\begin{aligned} \Delta\theta &= \frac{1}{2}\left((k_2x - \omega_2t) - (k_1x - \omega_1t)\right) \\ &= \frac{1}{2}(k_2x - k_1x - \omega_2t + \omega_1t) \\ &= \frac{1}{2}\left((k_2 - k_1)x - (\omega_2 - \omega_1)t\right) \\ &= \frac{1}{2}(k_2 - k_1)x - \frac{1}{2}(\omega_2 - \omega_1)t \end{aligned}$$

Here, we substitute in $\Delta k = \frac{1}{2}(k_2 - k_1)$ and $\Delta\omega = \frac{1}{2}(\omega_2 - \omega_1)$ and we get:

$$\Delta\theta = \Delta kx - \Delta\omega t$$

Substituting these back into the original wave equation, we get:

$$\begin{aligned} e^{i\theta_1} + e^{i\theta_2} &= 2\cos(\Delta\theta)\,e^{i\bar{\theta}} \\ &= 2\cos(\Delta kx - \Delta\omega t)\,e^{i\left(\bar{k}x-\bar{\omega}t\right)} \\ &= 2\cos(\Delta kx - \Delta\omega t)\,e^{i\bar{k}x-i\bar{\omega}t} \end{aligned}$$

The $e^{i\bar{k}x-i\bar{\omega}t} = e^{i\left(\bar{k}x-\bar{\omega}t\right)} = \cos\left(\bar{k}x - \bar{\omega}t\right) + i\sin\left(\bar{k}x - \bar{\omega}t\right)$ term describes the phase, and the

$\cos(\Delta kx - \Delta\omega t)$ term describes the wave packet. From the general wave equation, $f(x) = \sin(kx)$, for a traveling wave we have $f(x - vt) = \sin(k(x - vt)) = \sin\left(k\left(x - \left(\frac{\omega}{k}\right)t\right)\right) = \sin(kx - \omega t)$. From this, we can see that the general wave equation is analogous to the wave equations for the phase and the group, and that phase velocity is $v_\phi = \frac{\bar{\omega}}{\bar{k}}$ and the wave packet (group) velocity is $v_g = \frac{\Delta w}{\Delta k}$.

The beat frequency is $f_{beat} = \frac{\Delta\omega}{2\pi}$.

Part c

> Interpret part (a) as a specific case of part (b).

Part (a) has two waves with $k = k_1 = -k_2$, so $\Delta k = k$, and $\bar{k} = 0$. Also, $\omega = \omega_1 = \omega_2$, so we have $\Delta\omega = 0$ and $\bar{\omega} = \omega$. This gives us:

$$\begin{aligned} Ae^{ikx-i\omega t} + Ae^{-ikx-i\omega t} &= 2A\cos(kx - 0t)\, e^{i0x-i\omega t} \\ &= 2A\cos(kx)\, e^{-iwt} \end{aligned}$$

Note: For standing waves, this gives that $v_g = 0$, but $v_\phi = \infty$. However, the infinite phase velocity only indicates that the wave propagates across all points in space simultaneously, which is the case for a standing wave.

Part d

> In the limit that the two frequencies are very close together, show that the group velocity is $v_g = d\omega/dk$ for this case. The formula holds in general.

We have, for any constant v_0, that $v_0 = \frac{\omega}{k}$. We see, then, that as k changes, ω must change to preserve the ratio. Similarly, we have some $v_1 = \frac{\omega_1}{k_1}$ and some $v_2 = \frac{\omega 2}{k_2}$, so as $k_1 \to k_2$, we must also have $\omega_1 \to \omega_2$. Since $\Delta\omega = \frac{1}{2}(\omega_2 - \omega_1)$ and $\Delta k = \frac{1}{2}(k_2 - k_1)$, we have:

$$\begin{aligned} \lim_{\omega_2\to\omega_1} \Delta\omega &= \lim_{\omega_2\to\omega_1} \frac{1}{2}(\omega_2 - \omega_1) \\ &= d\omega \end{aligned}$$

and

$$\begin{aligned} \lim_{k_2\to k_1} \Delta k &= \lim_{k_2\to k_1} \frac{1}{2}(k_2 - k_1) \\ &= dk \end{aligned}$$

So, if we take $\lim_{k_2\to k_1} v_g = \frac{\Delta\omega}{\Delta k}$, we also know that $\omega_2 \to \omega_1$, and we end up with $v_g = \frac{d\omega}{dk}$.

Problem 3

> Wave packets. Experiment with the Java applet on the webpage: http://phet.colorado.edu/simulations/sims.php?sim=Fourier_Making_Waves

Part a

> Sketch the series of waves formed by adding each of the coefficients in succession: $A_1 = 1.27$, $A_3 = 0.52$, $A_5 = 0.25$, $A_7 = 0.18$, $A_9 = 0.14$, $A_{11} = 0.11$. Why are all of the even terms 0?

The wave forms can be seen in figures 1 through 6.

To simulate a square wave like this, all the even terms must be zero because a non-zero even term would go to zero in the middle fo the square wave form, causing the wave to drop to zero in the middle of the square wave.

Part b

> Determine the coefficients needed to reproduce the "inverted parabola" wave in figure 7:

The coefficients $A_1 = 0.56$, $A_3 = 0.62$, $A_5 = 0.25$, $A_7 = 0.07$, $A_9 = 0.01$ gave a good very good "inverted parabola," as seen in figure 8.

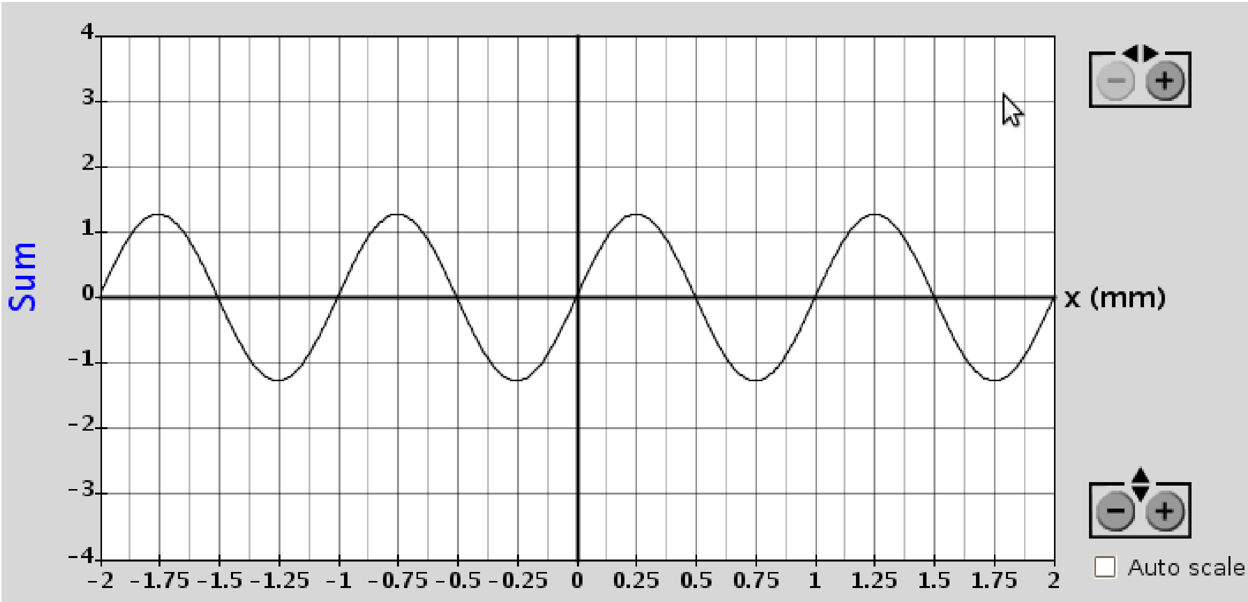

Figure 1: Wave with $A_1 = 1.27$

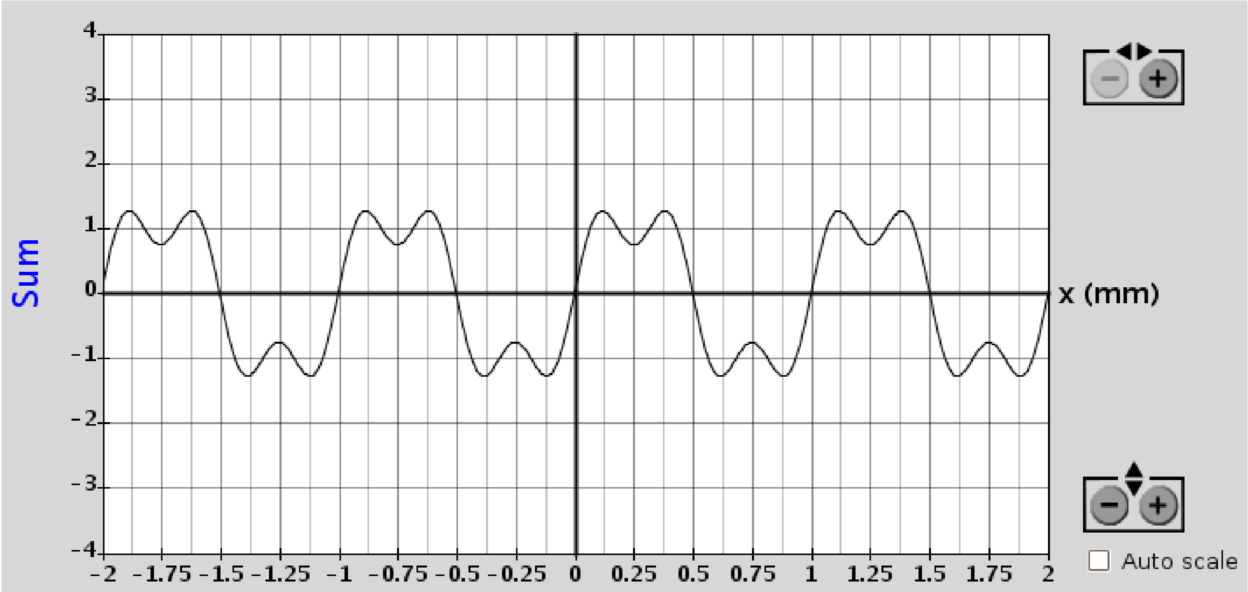

Figure 2: Wave with $A_1 = 1.27$ and $A_3 = 0.52$

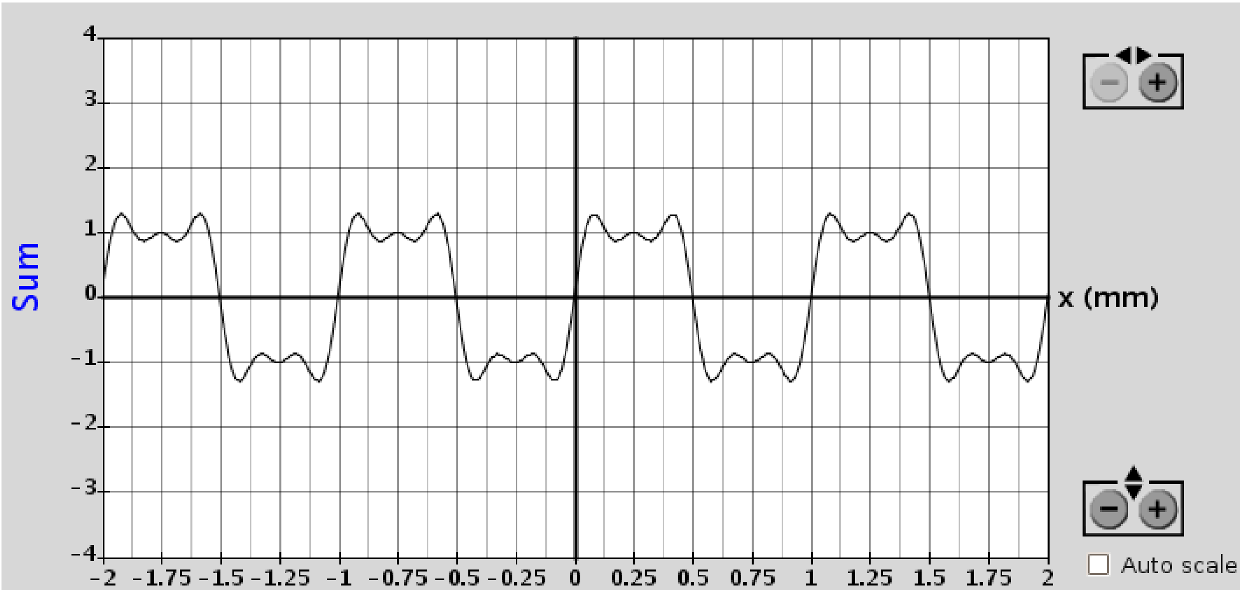

Figure 3: Wave with $A_1 = 1.27$, $A_3 = 0.52$ and $A_5 = 0.25$

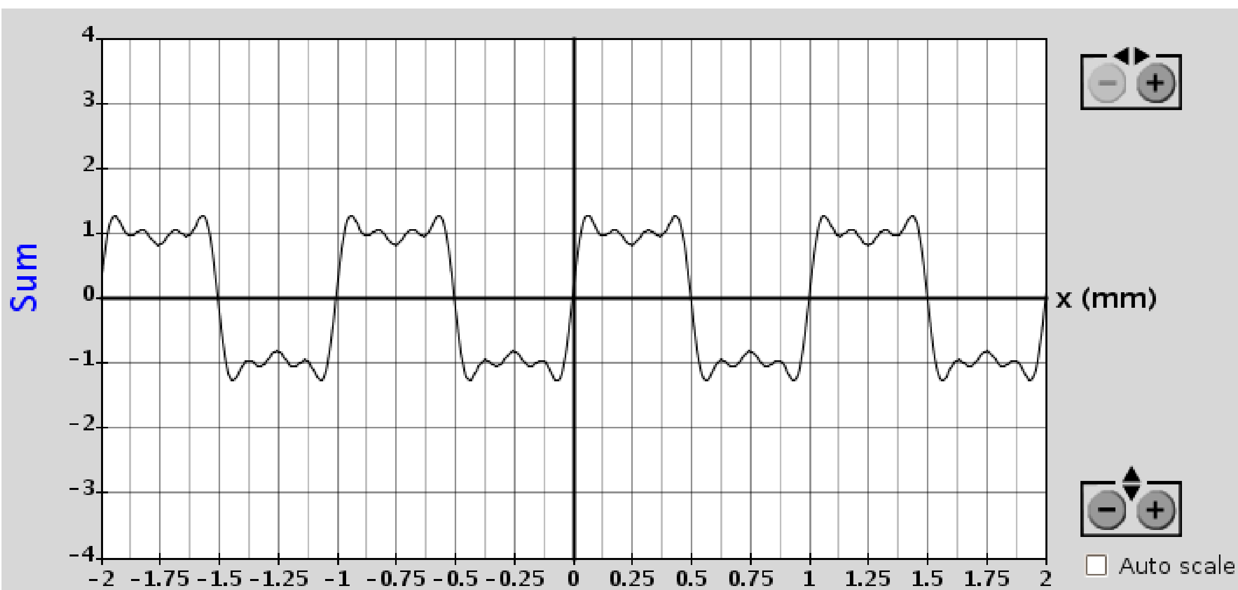

Figure 4: Wave with $A_1 = 1.27$, $A_3 = 0.52$, $A_5 = 0.25$, $A_7 = 0.18$

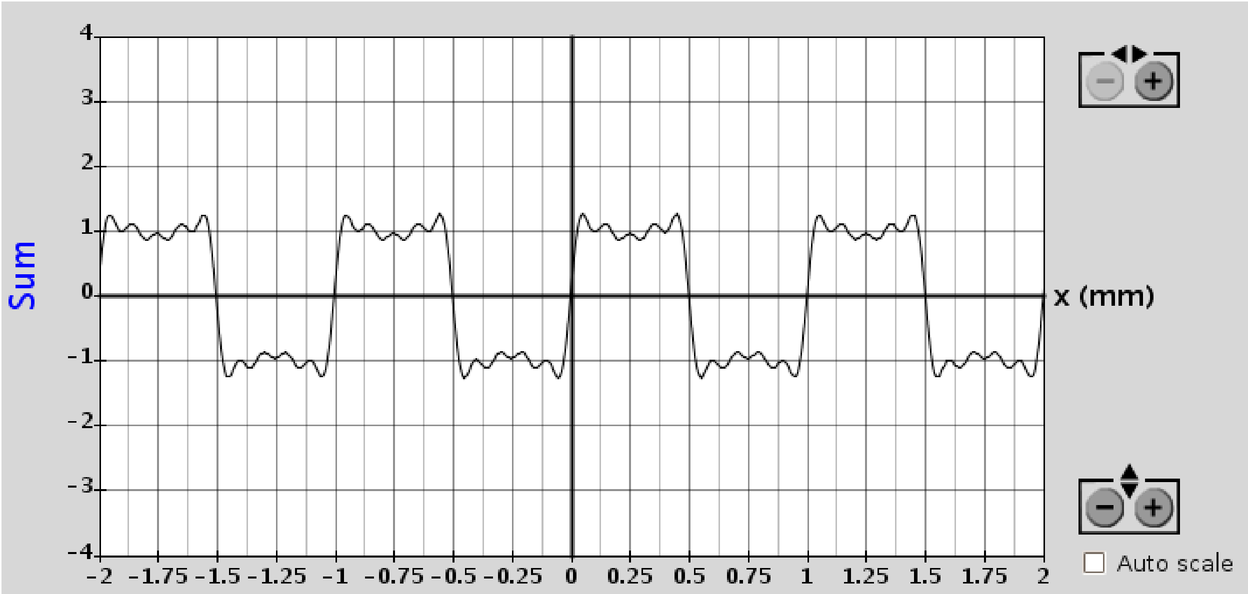

Figure 5: Wave with $A_1 = 1.27,\ A_3 = 0.52,\ A_5 = 0.25,\ A_7 = 0.18,\ A_9 = 0.14$

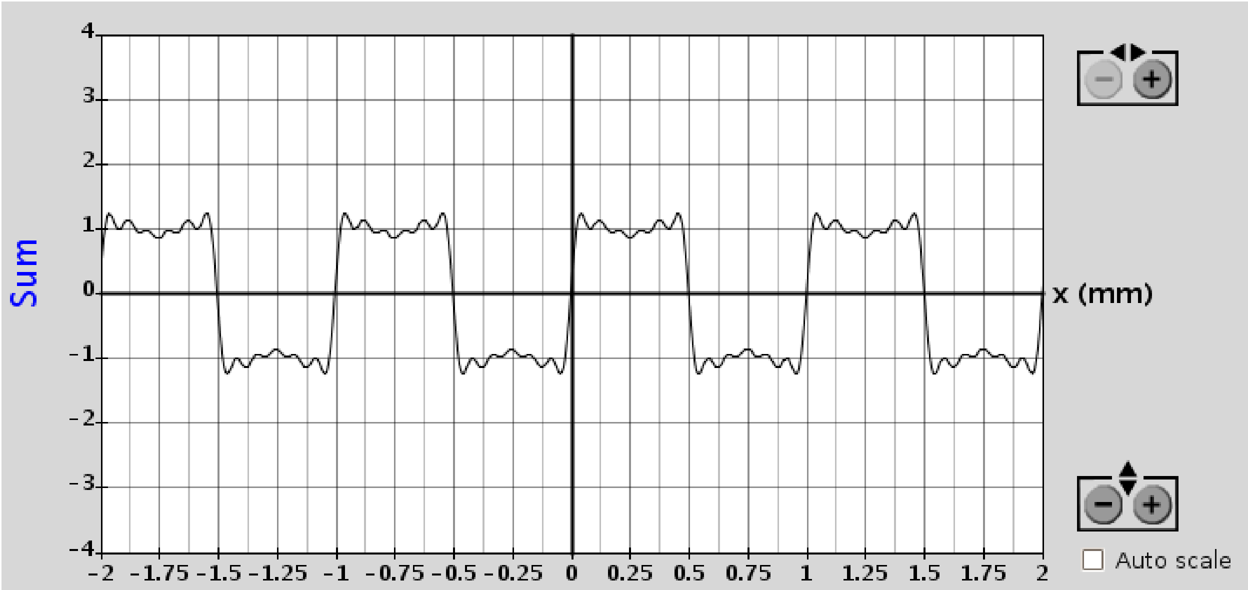

Figure 6: Wave with $A_1 = 1.27,\ A_3 = 0.52,\ A_5 = 0.25,\ A_7 = 0.18,\ A_9 = 0.14,\ A_{11} = 0.11$

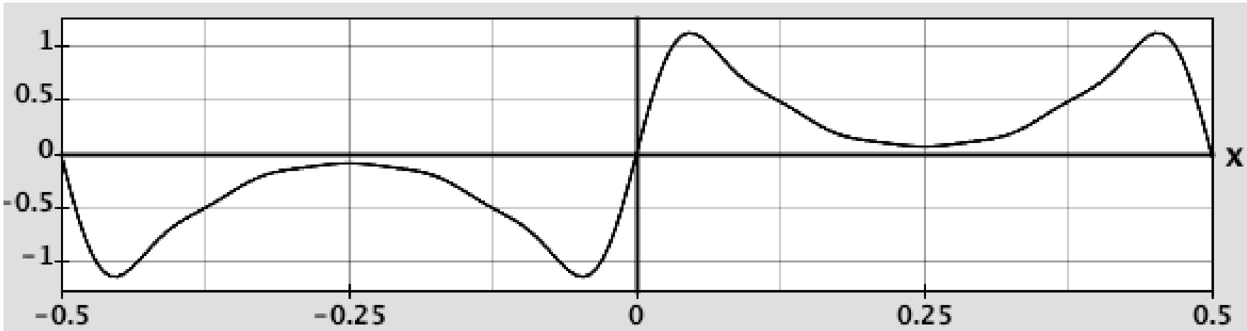

Figure 7: "Inverted parabola" wave for problem 3 part (b)

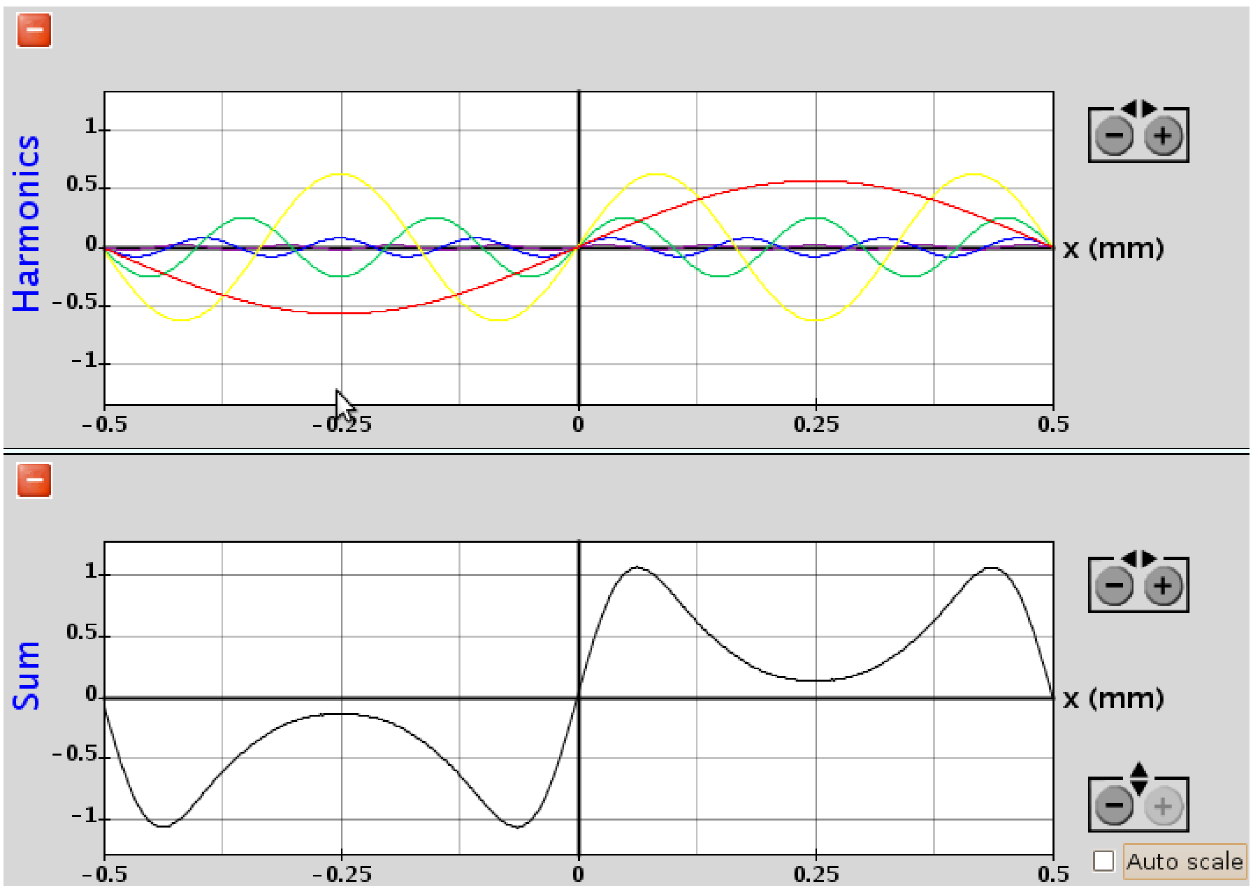

Figure 8: "Inverted Parabola"

Part c

> Solve a "Wave Game" puzzle of level 8 or higher and attach the printed result (or email me a screendump).

Figure 9 shows a completed level 10 game.

Part d

> In the "Discrete to Continuous" panel, explain what the three plots represent. Describe the variables k_1, λ_1, k_0, σ_k, and σ_x.

The Amplitude plot shows the distribution of frequencies used in constructing the final wave form. The Components plot shows all of the individual functions superimposed in the same plot. The Sum plot shows the sum of the functions drawn in the Components plot. The variable k_1 gives the magnitude of the separation between the frequencies of the sinusoidal functions used to generate the final wave function. It also represents the fundamental frequency of the spatial function. λ_1 gives the separation of the wave packet. k_0 gives the most probable angular frequency in the angular frequency distribution (the phase frequency). Each k_i also describes the momentum of the waves composing the wave packet. σ_k describes the uncertainty inherent in the estimated angular frequency, and therefore velocity and momentum of the wave packet, and σ_x describes uncertainty in the position of the wave packet.

Part e

> What is the effect of changing k_1 on the resulting amplitude distribution and wave packet? What happens as k_1 goes to zero?

Decreasing k_1 increases the number of samples represented in the amplitude distribution and increases the period of the wave equation. When k_1 goes to zero, the amplitude distribution becomes continuous, and the period of the wave equation becomes infinite, leaving a single wave packet.

Part f

> Repeat for k_0 and σ_k.

Changing k_0 changes the maximal value of the amplitude distribution, which shifts the distribution towards or away from zero. Smaller values of k_0 cause fewer phase oscillations in the wave packet (a lower phase frequency), while larger values of k_0 cause more phase oscillations (a higher phase frequency).

Changing σ_k (and, thus, σ_x in an inverse relationshiop) changes the width of the amplitude distribution and the width of the wave packet. Smaller values of σ_k yield smaller amplitude distributions, allowing us to know more precisely the momentum of the wave packet, but preventing us from knowing much about the location of the wave packet. Increasing σ_k decreases our knowledge about the actual momentum of the wave packet, but increases our knowledge about the location of the wave packet.

Part g

> Explain what Heisenberg uncertainty principle has to do with the frequency components of a wave packet.

The Heisenberg uncertainty principle states that $\Delta k \Delta x \sim 1$ (or, using the java applet notation, $\sigma_k \sigma_x \sim 1$), and that $\Delta\omega\Delta t \sim 1$. So, the more we know about the spacial frequency for a wave packet, the less we know about the position of the wave packet, and the more we know about the angular frequency, the less we know about time. More, because $p = \hbar k$ and $E = \hbar\omega$, we also have $\Delta x \Delta p \geq \hbar/2$ and $\Delta E \Delta t \geq \hbar/2$, telling us that the more we know about the momentum of a wave packet, the less we know about the position of that wave packet, and the more we know about the energy of a wave packet, the less we know about the time of that wave packet.

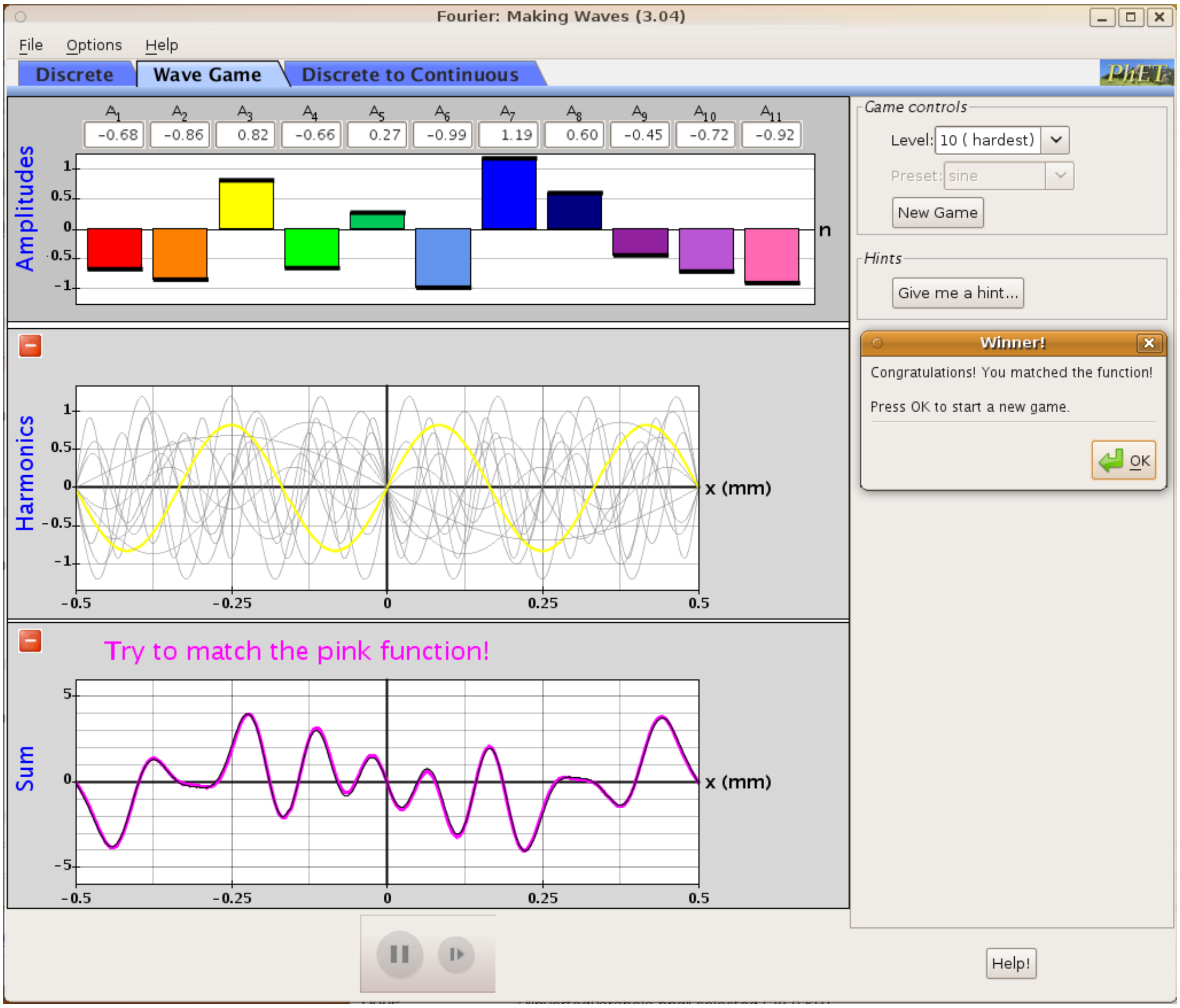

Figure 9: "Inverted Parabola"

Problem 5.16

Information is transmitted along a cable in the form of short electric pulses at 100,000 pulses/s.

Part a

What is the longest duration of the pulses such that they do not overlap?

The frequency, f, of the electric pulses is 100,000 Hz. The signal period is $1/f = 0.00001\,\mathrm{s}$. If the electrical pulse has a duration longer than this, then it will overlap the next electric pulse.

Part b

What is the range of frequencies to which the receiving equipment must respond for this duration?

The minimum frequency is 100,000 Hz, otherwise it is impossible to have a complete waveform in the specified period. The frequency range can be calculated by using the uncertainty principle, $\Delta\omega\Delta t \sim 1$, or in terms of frequency as $\Delta f \Delta t \sim \frac{1}{2\pi}$. So we have $\Delta f \sim \frac{1}{2\pi\Delta t} = \frac{1}{2\pi\cdot 0.00001}$ or $\Delta f \sim 15915\,\mathrm{Hz}$. So the maximum frequency is 115,915 Hz.

Problem 5.25

The wave function describing a state of an electron confined to move along the x axis is given at time zero by

$$\Psi(x,0) = Ae^{-x^2/4\sigma^2}$$

Part a

Find the probability of finding the electron in a region dx centered at $x = 0$.

The probability distribution for wave functions in general is given by $P(x)\,dx = |\Psi|^2\,dx$. First, we should determine A in $\Psi(x,0)$. We can do this by normalizing $P(x)$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} P(x)\,dx \\ 1 &= \int_{-\infty}^{\infty} \left|Ae^{-x^2/4\sigma^2}\right|^2 dx \\ 1 &= \int_{-\infty}^{\infty} A^2 e^{-\frac{1}{2\sigma^2}x^2} dx \\ 1 &= A^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} dx \end{aligned}$$

Setting $u = \frac{1}{2\sigma^2}$, using the integerals in [Tipler & Llewellyn, p. AP-16], and recognizing that e^{-x^2} is an even function we get:

$$\begin{aligned} 1 &= A^2 \int_{-\infty}^{\infty} e^{-ux^2} dx \\ 1 &= A^2 \sqrt{\frac{\pi}{u}} \\ 1 &= A^2 \sqrt{2\sigma^2\pi} \\ \frac{1}{\sqrt{2\sigma^2\pi}} &= A^2 \end{aligned}$$

Finally, we have:

$$\begin{aligned} P(x)\,dx &= \left|Ae^{-0^2/4\sigma^2}\right|^2 dx \\ &= |A\cdot 1|^2\,dx \\ &= A^2 dx \\ &= \frac{1}{\sqrt{2\sigma^2\pi}}\,dx \end{aligned}$$

Part b

Find the probability of finding the electron in a region dx centered at $x = \sigma$.

At $x = \sigma$, we have:

$$P(x)\,dx = \left|Ae^{-\sigma^2/4\sigma^2}\right|^2 dx$$

$$
\begin{aligned}
&= \left|Ae^{-1/4}\right|^2 dx \\
&= A^2 e^{-1/2} dx \\
&= \frac{1}{\sqrt{2\sigma^2 \pi e}}\, dx \\
&= \frac{0.6065}{\sqrt{2\sigma^2 \pi}}\, dx
\end{aligned}
$$

Part c

> Find the probability of finding the electron in a region dx centered at $x = 2\sigma$.

At $x = 2\sigma$ we have:

$$
\begin{aligned}
P(x)\, dx &= \left|Ae^{-(2\sigma)^2/4\sigma^2}\right|^2 dx \\
&= A^2 \left|e^{-4\sigma^2/4\sigma^2}\right|^2 dx \\
&= A^2 e^{-2} dx \\
&= \frac{1}{e^2\sqrt{2\sigma^2 \pi}}\, dx \\
&= \frac{0.1353}{\sqrt{2\sigma^2 \pi}}\, dx
\end{aligned}
$$

Part d

> Where is the electron most likely to be found?

The electron is most likely to be found at $x = 0$.

Problem 5.27

> If an excited state of an atom is known to have a lifetime of 10^{-7}s, what is the uncertainty in the energy of photons emitted by such atoms in the spontaneous decay to the ground state?

We can take the lifetime, τ, to be a measure of the time available to determine the energy of the state. The uncertainty in the energy corresponding to this time is:

$$
\begin{aligned}
\Delta E &\geq \frac{\hbar}{2\tau} \\
&\geq \frac{6.5821 \times 10^{-16}\text{eV}\cdot\text{s}}{(2)\left(10^{-7}\text{s}\right)} \\
&\geq 3.2911 \times 10^{-9}\text{eV}
\end{aligned}
$$

Problem 5.37

> Show that the relation $\Delta p_s \Delta s > \hbar$ can be written $\Delta L \Delta \phi > \hbar$ for a particle moving in a circle about the z axis, where p_s is the linear momentum tangential to the circle, s is the arc length, and L is the angular momentum. How well can the angular position of the electron be specified in the Bohr atom?

We know that $p_s = mv$, that $s = r\phi$, and that $L = mvr$. So, substituting in we have:

$$
\begin{aligned}
\hbar &< \Delta p_s \Delta s \\
&< \Delta(mv)\,\Delta(r\phi)
\end{aligned}
$$

We can assume that m is constant in this case, and if the particle is moving about a circle, then we can also assume that r, the radius of the circle, is fixed as well. So we have:

$$
\begin{aligned}
\hbar &< \Delta(mv)\,\Delta(r\phi) \\
&< \Delta mrv \Delta\phi \\
&< \Delta L \Delta \phi
\end{aligned}
$$

The uncertainty in the angular position $\Delta\phi$, then, will always be:

$$\Delta\phi > \frac{\hbar}{\Delta L}$$

But, $L = n\hbar$, so $\Delta L = \Delta n\hbar$, since $\hbar$ is constant. This leaves us with:

$$
\begin{aligned}
\Delta\phi &> \frac{\hbar}{\Delta n\hbar} \\
&> \frac{1}{\Delta n}
\end{aligned}
$$

Problem 5.41

> Using the relativistic expression $E^2 = p^2c^2 + m^2c^4$:

Part a

> Show that the phase velocity of an electron wave is greater than c.

We know that $v_p = E/p$, and from special relativity we have $E = \sqrt{p^2c^2 + m^2c^4} = \gamma mc^2$ and $p = \gamma mv$. So, we have:

$$\begin{aligned} v_p &= \frac{\gamma mc^2}{\gamma mv} \\ &= \frac{c^2}{v} \end{aligned}$$

But, $v < c$, so $\frac{v}{v} = 1 < \frac{c}{v}$ and $c < \frac{c^2}{v} = v_p$, i.e. the magnitude of the phase velocity for an electron wave is always greater than c.

Part b

> Show that the group velocity of an electron wave equals the particle velocity of the electron.

We know that the group velocity is $v_g = dE/dp$. Calculating the derivative of E, we have:

$$\begin{aligned} \frac{dE}{dp} &= \frac{d}{dp}\left(p^2c^2 + m^2c^4\right)^{1/2} \\ &= \frac{1}{2}2c^2p\left(p^2c^2 + m^2c^4\right)^{-1/2} \\ &= \frac{c^2p}{(p^2c^2 + m^2c^4)^{1/2}} \\ &= \frac{c^2p}{E} \end{aligned}$$

But, we know (from special relativity) that $p = \gamma mv$ and $E = \gamma mc^2$, so:

$$\begin{aligned} \frac{dE}{dp} &= \frac{c^2p}{E} \\ &= \frac{c^2\gamma mv}{\gamma mc^2} \\ &= v \end{aligned}$$

So, $v_g = dE/dp = v$.