Solutions to HW#6

1. a) $\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}f(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho f' = \frac{1}{\rho}(f'+\rho f'') = \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)f$

$$\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho}f = \frac{1}{\sqrt{\rho}}\frac{\partial}{\partial\rho}\left(\frac{1}{2\sqrt{\rho}}f+\sqrt{\rho}f'\right) = \frac{1}{\sqrt{\rho}}\left(\frac{-1}{4\rho^{3/2}}f+\frac{1}{2\sqrt{\rho}}f' + \frac{1}{2\sqrt{\rho}}f' + \sqrt{\rho}f''\right)$$

$$= \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{4\rho^2}\right)f.$$

thus $\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} = \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho}+\frac{1}{4\rho^2}$

b) $\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 f' = \frac{1}{r^2}(2rf'+r^2f'') = \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)f$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}rf = \frac{1}{r}\frac{\partial}{\partial r}(f+rf') = \frac{1}{r}(f'+f'+rf'') = \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)f$$

thus $\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r$

2. a) $\hat{T}\Psi+\hat{V}\Psi = E\Psi$ $\qquad \hat{T} = \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\underbrace{\nabla^2}_{\text{see prob \#1.}}$

$$\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\cancelto{0}{\frac{\partial^2}{\partial z^2}}\right)\Psi + \cancelto{0}{V(r)}\Psi = E\Psi(\rho,\phi)$$

b) let $\Psi(\rho,\phi) = R(\rho)\,\Phi(\phi)$

$$\frac{-\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)R\Phi = E\cdot R\Phi$$

$$\frac{\cancel{\Phi(\phi)}\cdot\left(\rho^2\frac{\partial^2}{\partial\rho^2}+\rho\frac{\partial}{\partial\rho}\right)R}{\cancel{\Phi}\;R} + \frac{\frac{2\mu E}{\hbar^2}\cancel{R}\cancel{\Phi}\rho^2}{\cancel{R}\cancel{\Phi}} = \frac{-R\frac{\partial^2}{\partial\phi^2}\Phi(\phi)}{R\Phi} = \alpha$$

i) $\left\{\rho^2 \frac{\partial^2}{\partial\rho^2} + \rho\frac{\partial}{\partial\rho} + (k^2\rho^2 - \alpha)\right\} R(r) = 0$ where $E = \frac{\hbar^2 k^2}{2\mu}$

ii) $\left[\frac{\partial^2}{\partial\phi^2}\right] \Phi(\phi) = -\alpha\Phi$

c) let $\Phi = e^{im\phi}$ then $\frac{\partial^2}{\partial\phi^2}\Phi = -m^2\Phi$

so $\alpha = m^2$

boundary condition: $\Phi(0) = \Phi(2\pi)$ $1 = e^{im\phi}\big|^{2\pi} = e^{i\{2\pi, 4\pi, 6\pi, \dots\}}$

thus. m has to be an integer. ($m \in \mathbb{Z}$)

d) let $x = k\rho$ then $dx = k\,d\rho$ (k is a constant)

let $R(\rho) = J(k\rho) = J(x)$

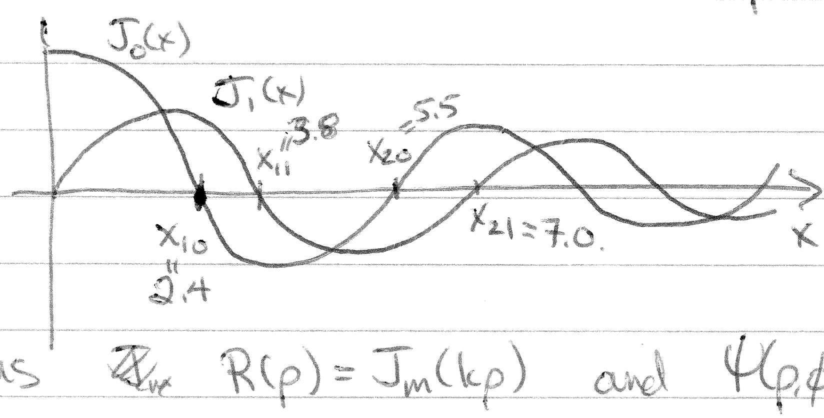
$$x^2 \frac{\partial^2}{\partial x^2} J(x) + x\frac{\partial}{\partial x} J(x) + (x^2 - m^2) J(x) = 0$$

$$x^2 J'' + x J' + (x^2 - m^2) J = 0$$

the solution is called the Bessel function

$J_m(x)$ of order m. of the first kind

(the other is discontinuous at $\rho = 0$).



thus $R(\rho) = J_m(k\rho)$ and $\Psi(\rho,\phi) = J_m(k\rho)e^{im\phi}$

e) The factor 'k' stretches $J(x)$ along the x-axis. so that the boundary condition $R(a) = 0$ can be satisfied.

ch7. #9. a) $n=3$ $\quad l=0,1,2$ $\quad$ #states

$l=0$ $\quad m=0$ $\quad 2$

$l=1$ $\quad m=1,0,-1$ $\quad 6$

$l=2$ $\quad m=2,1,0,-1,-2$ $\quad 10$

18 states.

#10 $l=4$ $\quad \cos\theta=\frac{L_z}{|L|}=\frac{4\hbar}{\sqrt{4(4+1)\hbar^2}}=\frac{2}{\sqrt{5}}$ $\quad \theta=26.6°$

b) $l=2$ $\quad \cos\theta=\frac{2\hbar}{\sqrt{2(2+1)\hbar^2}}=\frac{2}{\sqrt{6}}$ $\quad \theta=35.3°$

#15 $$\frac{dL}{dt}=\frac{d}{dt}r\times p=\cancel{v\times p}^{\,0}+r\times\frac{dp}{dt}=r\times F=\tau$$

$$=\vec{r}\times(-\nabla V)=-\vec{r}\times\left[\hat{r}\frac{\partial}{\partial r}V(r)+\cancel{\frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}V}^{\,0}+\cancel{\frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}V}^{\,0}\right]$$

$$=-\cancel{\vec{r}\times\hat{r}}^{\,0}\,V'(r)=0$$

#16. a) $l=3$ $\quad n>3=4,5\ldots$ $\quad E_4=\frac{-E_0}{4^2}=$ ~~-1.512 eV~~ -0.850 eV

$m=-3,-2,-1,0,1,2,3$

b) $l=4$ $\quad n>4=5,6,7\ldots$ $\quad E_5=\frac{-E_0}{5^2}=-0.544$ eV

$m=-4,-3,-2,-1,0,1,2,3,4$

c) $l=0$ $\quad n>0=1,2,3,\ldots$ $\quad E_1=-E_0=-13.6$ eV.

$m=0$.