HW #7 Solutions

(35)

a) $$\hat{T} = \frac{p^2}{2\mu} = \frac{(-i\hbar\nabla)^2}{2\mu} = \frac{-\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\underbrace{\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)}_{-\hat{L}^2/\hbar^2}\right)$$

$$= \underbrace{\frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}r}_{T_r \text{ (radial)}} + \underbrace{\frac{\hat{L}^2}{2\mu r^2}}_{T_\Omega \text{ (angular)}} = \frac{\hat{p}_r^2}{2\mu} + \frac{\hat{L}^2}{2I}$$

b) let $\psi = \frac{1}{r}u \cdot Y_{lm} = \frac{1}{r}u(r)\cdot Y_{lm}(\theta,\phi)$

$\hat{H}\psi = E\psi = (\hat{T}+\hat{V})\psi \qquad V = \frac{-Zk_e e^2}{r}$

$$\frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}r\left(\frac{1}{r}u\cdot Y_{lm}\right) + \frac{\hat{L}^2}{2\mu r^2}\left(\frac{1}{r}u\cdot Y_{lm}\right) - \frac{Zk_e e^2}{r}\left(\frac{1}{r}uY_{lm}\right) = E\left(\frac{1}{r}uY\right)$$

$$\frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}u\cdot \cancel{Y_{lm}} + \frac{\hbar^2 l(l+1)}{2\mu r^2}u\cdot \cancel{Y_{lm}} - \frac{Zk_e e^2}{r}u\cdot Y_{lm} = E\cdot u\cancel{Y_{lm}}$$

$$\underbrace{\frac{-\hbar^2}{2\mu}u''}_{\text{1-d: }\hat{T}_r} + \underbrace{\frac{\hbar^2 l(l+1)}{2\mu r^2}u}_{\hat{V}_{cent}} - \underbrace{\frac{Zk_e e^2}{r}u}_{\hat{V}_{coulomb}} = E\cdot u$$

c) $L = mvr$

$$F_{cent} = -\nabla V_{cent} = -\hat{r}\,V_{cent}'(r) = -\hat{r}\frac{\partial}{\partial r}\frac{L^2}{2\mu r^2} = \hat{r}\frac{L^2}{\mu r^3} = \hat{r}\,\mu\underbrace{\frac{v^2}{r}}_{a_{cent.}}$$

d) $$\left[\frac{-\hbar^2}{2\mu}\left(\frac{2Z\mu k_e e^2}{n\hbar^2}\right)^2\frac{\partial^2}{\partial\rho^2} + \frac{\hbar^2 l(l+1)}{2\mu\rho^2}\left(\frac{2Z\mu k_e e^2}{n\hbar^2}\right)^2 - \frac{Zk_e e^2\mu}{\rho}\left(\frac{2Z\mu k_e e^2}{n\hbar^2}\right)\right]u(\rho) = \frac{-Z^2\mu(k_e e^2)^2}{n^2\,2\hbar^2}u(\rho)$$

$$\left(\frac{\partial^2}{\partial\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{n}{\rho} - \frac{1}{4}\right)u(\rho) = 0.$$

e) $$((fgh)')' = (f'gh)' + (fg'h)' + (fgh')' = (f''gh + f'g'h + f'gh') + \ldots$$

$$= (f''gh + fg''h + fgh'') + 2(f'g'h + f'gh' + fg'h')$$

f) let $u = e^{-\rho/2} \rho^{l+1} L(\rho) = f g \cdot h$ $\quad f' = -\frac{1}{2} f \quad g' = (l+1) g/\rho$

$f'' = \frac{1}{4} f \quad g'' = \frac{l(l+1)}{\rho^2} g.$

$$u'' = \left(\frac{1}{4} f g h + f \cdot \frac{l(l+1)}{\rho^2} g h + f g \cdot L''\right) + 2\left(-\frac{1}{2} f \cdot \frac{l+1}{\rho} g \cdot L + -\frac{1}{2} f \cdot g \cdot L' + f \cdot \frac{l+1}{\rho} g \cdot L'\right)$$

$$u'' - \frac{l(l+1)}{\rho^2} u + \frac{n}{\rho} u - \frac{1}{4} u = \Big[\cancel{\tfrac{1}{4} u} + \cancel{\tfrac{l(l+1)}{\rho^2} u} + L'' f g$$

$$- \frac{l+1}{\rho} L \cdot f g - L' \cdot f g + \frac{2(l+1)}{\rho} L' \cdot f g - \cancel{\frac{l(l+1)}{\rho^2} u} + \frac{n}{\rho} u - \cancel{\frac{1}{4} u} = 0$$

multiply by $\frac{\rho}{fg}$:

$$\rho L'' + (\underbrace{2l+2-\rho}_{(2l+1)+1-\rho}) L' + (\underbrace{n-l-1}_{\kappa}) L = 0$$

$\alpha = 2l+1$

solution: $L = L_{n-l-1}^{(2l+1)}(\rho) = L_{\kappa}^{(2l+1)}(\rho)$ $\quad \kappa = 0, 1, 2 \ldots$

= # of radial nodes.

g) solutions:

$n = 1, \ l = 0$: $\quad \alpha = 1 \quad \kappa = 0 \quad L_{\kappa}^{(\alpha)} = 1$

$R_{nl}(r) = \frac{1}{r} u\left(r \cdot \frac{2Z}{n a_0}\right) r^{l+1} e^{-rZ/na_0} \cdot C_{nl}$ normalization

$R_{10} \propto r^0 e^{-rZ/a_0}$

$n = 2 \quad l = 0 \quad \alpha = 1 \quad \kappa = 1 \quad L_{\kappa}^{(\alpha)}(\rho) = -\rho + 2$

$R_{20} \propto \left(1 - \frac{Zr}{2a_0}\right) e^{-rZ/2a_0}$

$n = 2 \quad l = 1 \quad \alpha = 3 \quad \kappa = 0 \quad L_{\kappa}^{(\alpha)}(\rho) = 1$

$R_{21} \propto \frac{Zr}{2a_0} e^{-Zr/2a_0}$

$n = 3 \quad l = 0 \quad \alpha = 1 \quad \kappa = 2 \quad L_{\kappa}^{(\alpha)} = \frac{1}{2}[\rho^2 - 6\rho + 6]$

$R_{30} \propto \left(1 - \frac{2Zr}{3a_0} + \frac{2}{27}\left(\frac{Zr}{a_0}\right)^2\right) e^{-Zr/3a_0}$

$n = 3 \quad l = 1 \quad \alpha = 3 \quad \kappa = 1 \quad L_{\kappa}^{(\alpha)} = -\rho + 4$

$R_{31} \propto \left(1 - \frac{Zr}{6a_0}\right) e^{-Zr/3a_0}$

$n = 3 \quad l = 2 \quad \alpha = 5 \quad \kappa = 0 \quad L_{\kappa}^{(\alpha)} = 1$

$R_{32} \propto r^2 e^{-Zr/3a_0}$ $\quad$ same except normalization

g) $\int d^3r\,|\psi_{100}|^2 = \int r^2 dr \sin\theta d\theta d\phi \left(\frac{2}{\sqrt{a_0^3}} e^{-r/a_0}\right)^2 \cdot Y_{00}(\theta,\phi)$

$= \int_0^\infty \frac{4}{a_0^3} r^2 e^{-2r/a_0} dr \cdot \underbrace{\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \left(\frac{1}{\sqrt{4\pi}}\right)^2}_{= \frac{4\pi}{4\pi} = 1}$

$= \frac{4}{a_0^3}\left[r^2 \frac{a_0}{2} e^{-2r/a_0} - 2r\frac{a_0^2}{4} e^{-2r/a_0} + 2\frac{a_0^3}{8} e^{-2r/a_0}\right]_0^\infty = 1$

#26 $P(r) = r^2 R_{n\ell}^2(r)$ let $n=2$, $\ell=1$

$0 = \frac{dP}{dr} = \frac{d}{dr}\left(\left(1-\frac{r}{2a_0}\right)e^{-r/2a_0}\right)$

$= \frac{-1}{2a_0} e^{-r/2a_0} + \left(1-\frac{r}{2a_0}\right)\frac{-1}{2a_0} e^{-r/2a_0}$

$0 = \frac{-1}{a_0} + \frac{r}{4a_0^2}$ $r_{max} = 4a_0 = n^2 a_0$

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#29 in general, $\mu = \overset{\text{current}}{I} A = \frac{q\cdot\omega}{2\pi}\cdot\pi\langle r^2\rangle = \frac{q}{2}\omega\langle r^2\rangle_q$ averaged over charge distribution

$L = \mathcal{I}\omega = m\omega\langle r^2\rangle_m$ averaged over mass distribution.

moment of inertia

note: mass and charge do not have to be at the same place.

so $\mu = \frac{q}{2m}\frac{\langle r^2\rangle_q}{\langle r^2\rangle_m} L$ and $g = \frac{\langle r^2\rangle_q}{\langle r^2\rangle_m} = \frac{m\langle r^2\rangle_q}{\mathcal{I}}$

where $\mathcal{I} = m\langle r^2\rangle_m$

a) $g = \frac{R^2}{\frac{1}{2}R^2} = 2$

b) $g = \frac{R^2}{2/5 R^2} = 2.5$

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#39 a) $\vec{L} = \vec{L}_1 + \vec{L}_2 = \{0,1,2\}$ b) $\vec{S} = \vec{S}_1 + \vec{S}_2 = \{0,1\}$

c) $\vec{J} = \vec{L} + \vec{S}$ $j = \{0,1,2,3\}$

d) $J_1 = L_1 + S_1$ $j_1 \in \{\frac{1}{2}, 3/2\}$

e) $J = J_1 + J_2$ $j = \{0,1,2,3\}$ $\left[{}^1S_0, {}^3S_1, {}^1P_1, {}^3P_{2,1,0}, {}^1D_2\ {}^3D_{3,2,1}\right]$

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#44 $_6C: 1s^2\,2s^2\,2p^2$

5

$_8O: 1s^2\,2s^2\,2p^4$

$_{18}Ar: 1s^2\,2s^2\,2p^6\,3s^2\,3p^6$

#73 a) $\mu_J = \frac{\vec{\mu}\cdot\vec{J}}{J} = \frac{(\vec{\mu}_L + \vec{\mu}_S)\cdot(\vec{L}+\vec{S})}{J}$

$= -\frac{\mu_B}{\hbar J}(\vec{L}+2\vec{S})\cdot(\vec{L}+\vec{S})$

$= -\frac{\mu_B}{\hbar J}(L^2 + 3L\cdot S + 2S^2)$

$\mu_L = -\frac{g_L \mu_B \vec{L}}{\hbar} = -\frac{\mu_B L}{\hbar}$

$\mu_S = -\frac{g_S \mu_B \vec{S}}{\hbar} = -2\frac{\mu_B \vec{S}}{\hbar}$

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b) $J^2 = (L+S)^2 = L^2 + 2L\cdot S + S^2$

$L\cdot S = \frac{1}{2}(J^2 - L^2 - S^2)$

c) $\mu_z = \frac{J_z}{J}\cdot\frac{-\mu_B}{\hbar J}\left(L^2 - \frac{3}{2}(J^2-L^2-S^2) + 2S^2\right)$

$= -\frac{\mu_B J_z}{2\hbar^2 J^2}(3J^2 - L^2 - S^2) = -\mu_B\left(1 + \frac{J^2+S^2-L^2}{2J^2}\right)\frac{J_z}{\hbar}$