# Problem 1

[35 pts] Hydrogen radial wavefunctions: the hydrogen potential is $V(r) = -Zk_e e^2/r$.

## Part a

Using the Lapacian $\nabla^2$ in spherical coordinates, show that

$$\hat{T} = \frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\hat{L}^2}{2\mu r^2}$$

where the second term represents rotational kinetic energy, with

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Write Schrödinger's equation for $\psi(r,\theta,\phi)$ of the hydrogen atom using this form of $\hat{T}$.

Similar to problem (2) part (a) in homework set #6, we start with the multi-dimensional time-independent Schrödinger's equation and substitute in the Lapacian in spherical coordinates:

$$\begin{aligned}
\hat{H}\psi &= E\psi \\
\hat{T}\psi + \hat{V}\psi &= E\psi \\
-\frac{\hbar^2}{2\mu}\nabla^2\psi + \hat{V}\psi &= E\psi \\
-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right)\psi + \hat{V}\psi &= E\psi
\end{aligned}$$

So now we must show that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}r$, so:

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) &= \frac{1}{r}\frac{\partial^2}{\partial r^2}r \\
\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f(r) &= \frac{1}{r}\frac{\partial^2}{\partial r^2}rf(r) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f(r) &= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}rf(r)\right) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f(r) &= \frac{1}{r}\frac{\partial}{\partial r}\left(rf'(r) + f(r)\right) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f(r) &= \frac{1}{r}\left(rf''(r) + f'(r) + f'(r)\right) \\
\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)f(r) &= f''(r) + \frac{2}{r}f'(r) \\
\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}
\end{aligned}$$

So, we can replace $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$ with $\frac{1}{r}\frac{\partial^2}{\partial r^2}r$.

$$\begin{aligned}
-\frac{\hbar^2}{2\mu}\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right)\psi + \hat{V}\psi &= E\psi \\
\left(\frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{-\hbar^2}{2\mu r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\right)\psi + \hat{V}\psi &= E\psi
\end{aligned}$$

If $\hat{L}^2 = -\hbar^2 \left( \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$, then we have:

$$\begin{aligned} \frac{-\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hat{L}^2}{2\mu r^2} \psi + \hat{V}\psi &= E\psi \\ \hat{T}\psi + \hat{V}\psi &= E\psi \end{aligned}$$

Schrödinger's equation is, then:

$$\left( \frac{-\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hat{L}^2}{2\mu r^2} \right) \psi(r,\theta,\phi) + \hat{V}\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

But $\hat{V} = V(r) = -Zk_e e^2/r$, so

$$\left( \frac{-\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hat{L}^2}{2\mu r^2} \right) \psi(r,\theta,\phi) - \frac{Zk_e e^2}{r}\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

## Part b

Make the substitution $\psi(r,\theta,\phi) = \frac{1}{r}u(r)Y_{lm}(\theta,\phi)$. The factor $\frac{1}{r}$ takes into account the spreading out of the wave function as it gets farther from the origin. Use the eigenvalue of $Y_{lm}$,

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$$

to simplify the equation.

We start with:

$$\left( \frac{-\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hat{L}^2}{2\mu r^2} \right) \psi(r,\theta,\phi) - \frac{Zk_e e^2}{r}\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

Substituting in $\psi(r,\theta,\phi) = \frac{1}{r}u(r)Y_{lm}(\theta,\phi)$, we get:

$$\begin{aligned} E\frac{1}{r}u(r)Y_{lm}(\theta,\phi) &= \left( \frac{-\hbar^2}{2\mu} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hat{L}^2}{2\mu r^2} \right) \frac{1}{r}u(r)Y_{lm}(\theta,\phi) - \frac{Zk_e e^2}{r}\frac{1}{r}u(r)Y_{lm}(\theta,\phi) \\ &= \frac{-\hbar^2}{2\mu}\frac{1}{r}Y_{lm}(\theta,\phi)\frac{\partial^2}{\partial r^2}\frac{r}{r}u(r) + \frac{u(r)}{2\mu r^3}\hat{L}^2 Y_{lm}(\theta,\phi) - \frac{Zk_e e^2}{r}\frac{1}{r}u(r)Y_{lm}(\theta,\phi) \\ &= \frac{-\hbar^2}{2\mu}\frac{1}{r}Y_{lm}(\theta,\phi)\frac{\partial^2}{\partial r^2}u(r) + \frac{u(r)}{2\mu r^3}\hbar^2 l(l+1)Y_{lm}(\theta,\phi) - \frac{Zk_e e^2}{r}\frac{1}{r}u(r)Y_{lm}(\theta,\phi) \\ E\frac{1}{r}u(r) &= \frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}u(r) + \frac{u(r)}{2\mu r^3}\hbar^2 l(l+1) - \frac{Zk_e e^2}{r}\frac{1}{r}u(r) \\ Eu(r) &= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}u(r) + \frac{u(r)}{2\mu r^2}\hbar^2 l(l+1) - \frac{Zk_e e^2}{r}u(r) \end{aligned}$$

## Part c

Show that the result looks like the Schrödinger equation for $u(r)$ in one dimension with a centrifugal potential $V_c(r) = \hbar^2 l(l+1)/2\mu r^2$ in addition to the Colomb potential. Compare with the potential of the centrifugal force $F = ma_c = mv^2/r$ using $L = mvr$.

If $V_c(r) = \hbar^2 l\left(l+1\right)/2\mu r^2$, then we can replace $\hbar^2 l\left(l+1\right)/2\mu r^2$:

$$Eu\left(r\right) = \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}u\left(r\right) + V_c\left(r\right)u\left(r\right) - \frac{Zk_e e^2}{r}u(r)$$

If $V_c\left(r\right) = \hbar^2 l\left(l+1\right)/2\mu r^2$, then we should be able to find $F = -\frac{d}{dr}\left(\hbar^2 l\left(l+1\right)/2\mu r^2\right)$. So:

$$\begin{aligned}
F &= -\frac{d}{dr}\left(\frac{\hbar^2 l\left(l+1\right)}{2\mu r^2}\right)\\
F &= -\frac{\hbar^2 l\left(l+1\right)}{2\mu}\frac{d}{dr}\left(r^{-2}\right)\\
F &= -\frac{\hbar^2 l\left(l+1\right)}{2\mu} - 2r^{-3}\\
F &= \frac{\hbar^2 l\left(l+1\right)}{\mu}r^{-3}\\
F &= \frac{L^2}{\mu r^3}\\
F &= \frac{\left(\mu v r\right)^2}{\mu r^3}\\
F &= \frac{\mu v^2}{r}
\end{aligned}$$

So, we can see that the derivative of $V_c\left(r\right)$ gives us an apparent centrifugal force.

## Part d

Make the substitutions

$$\begin{aligned}
u(r) &= U(\rho)\,\text{where}\,\rho = \frac{2r}{r_n}\\
r_n &= \frac{na_0}{Z}\,\text{where}\,a_0 = \frac{\hbar^2}{\mu k_e e^2}\\
E_n &= \frac{-Z^2 E_0}{n^2}\,\text{where}\,E_0 = \frac{\mu k_e^2 e^4}{2\hbar^2}
\end{aligned}$$

to obtain the dimensionless equation

$$\left(\frac{\partial^2}{\partial \rho^2} - \frac{l\left(l+1\right)}{\rho^2} + \frac{n}{\rho} - \frac{1}{4}\right)U(\rho) = 0$$

Starting with:

$$Eu\left(r\right) = \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}u\left(r\right) + \frac{u\left(r\right)}{2\mu r^2}\hbar^2 l\left(l+1\right) - \frac{Zk_e e^2}{r}u\left(r\right)$$

and substituting in $u(r) = U(\rho)$, where $\rho = \frac{2r}{r_n}$, and recognizing that $r = \rho r_n/2$, we get:

$$\begin{aligned}
EU\left(\rho\right) &= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}U\left(\rho\right) + \frac{U\left(\rho\right)}{2\mu r^2}\hbar^2 l\left(l+1\right) - \frac{Zk_e e^2}{r}U\left(\rho\right)\\
&= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}U\left(\rho\right) + \frac{U\left(\rho\right)}{2\mu\left(\frac{\rho r_n}{2}\right)^2}\hbar^2 l\left(l+1\right) - \frac{2Zk_e e^2}{\rho r_n}U\left(\rho\right)
\end{aligned}$$

$$\begin{aligned} &= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}U(\rho) + \frac{U(\rho)}{2\mu\left(\frac{\rho r_n}{2}\right)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho r_n}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}U(\rho) + \frac{2U(\rho)}{\mu(\rho r_n)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho r_n}U(\rho) \end{aligned}$$

We also have that $\frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho} = \frac{\partial}{\partial r}$, and that

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho}\right)$$

But, we know that $\frac{\partial\rho}{\partial r} = \frac{\partial}{\partial r}\left(\frac{2r}{r_n}\right) = \frac{2}{r_n}$, so this becomes:

$$\begin{aligned} \frac{\partial^2}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{2}{r_n}\frac{\partial}{\partial\rho}\right) \\ &= \frac{2}{r_n}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\rho}\right) \\ &= \frac{2}{r_n}\left(\frac{\partial\rho}{\partial r}\frac{\partial}{\partial\rho}\right)\left(\frac{\partial}{\partial\rho}\right) \\ &= \left(\frac{2}{r_n}\right)^2\frac{\partial^2}{\partial\rho^2} \\ &= \frac{4}{r_n^2}\frac{\partial^2}{\partial\rho^2} \end{aligned}$$

So, substituting this in, we get:

$$\begin{aligned} EU(\rho) &= \frac{-\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2}U(\rho) + \frac{2U(\rho)}{\mu(\rho r_n)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho r_n}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{4}{r_n^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{2U(\rho)}{\mu(\rho r_n)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho r_n}U(\rho) \end{aligned}$$

Now, since $r_n = \frac{na_0}{Z}$ , $a_0 = \frac{\hbar^2}{\mu k_e e^2}$, $E_n = \frac{-Z^2E_0}{n^2}$, and $E_0 = \frac{\mu k_e^2 e^4}{2\hbar^2}$.

$$\begin{aligned} EU(\rho) &= \frac{-\hbar^2}{2\mu}\frac{4}{r_n^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{2U(\rho)}{\mu(\rho r_n)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho r_n}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{4}{\left(\frac{na_0}{Z}\right)^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{2U(\rho)}{\mu\left(\rho\frac{na_0}{Z}\right)^2}\hbar^2 l(l+1) - \frac{2Zk_e e^2}{\rho\frac{na_0}{Z}}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{Z^2 4}{(na_0)^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{2ZU(\rho)}{\mu(\rho na_0)^2}\hbar^2 l(l+1) - \frac{2Z^2k_e e^2}{\rho na_0}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{Z^2 4}{\left(n\frac{\hbar^2}{\mu k_e e^2}\right)^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{2Z^2U(\rho)}{\mu\left(\rho n\frac{\hbar^2}{\mu k_e e^2}\right)^2}\hbar^2 l(l+1) - \frac{2Z^2k_e e^2}{\rho n\frac{\hbar^2}{\mu k_e e^2}}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{\left(\mu k_e e^2\right)^2 Z^2 4}{\left(n\hbar^2\right)^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{\left(\mu k_e e^2\right)^2 2ZU(\rho)}{\mu\left(\rho n\hbar^2\right)^2}\hbar^2 l(l+1) - \frac{2\mu e^2 Z^2 k_e^2 e^2}{\rho n\hbar^2}U(\rho) \\ &= \frac{-\hbar^2}{2\mu}\frac{\mu^2 k_e^2 e^4 Z^2 4}{n^2\hbar^4}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{\mu^2 k_e^2 e^4 2Z^2 U(\rho)}{\mu\rho^2 n^2\hbar^4}\hbar^2 l(l+1) - \frac{2\mu k_e^2 e^4 Z^2}{\rho n\hbar^2}U(\rho) \\ &= \frac{-1}{2}\frac{\mu k_e^2 e^4 Z^2 4}{n^2\hbar^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{\mu k_e^2 e^4 4Z^2 U(\rho)}{2\rho^2 n^2\hbar^2}l(l+1) - \frac{4\mu k_e^2 e^4 Z^2}{2\rho n\hbar^2}U(\rho) \end{aligned}$$

$$\begin{aligned}
&= -\frac{E_0 Z^2 4}{n^2}\frac{\partial^2}{\partial\rho^2}U(\rho) + \frac{E_0 Z^2 4U(\rho)}{\rho^2 n^2}l(l+1) - \frac{4E_0 Z^2 n}{\rho n^2}U(\rho) \\
&= 4E_n\frac{\partial^2}{\partial\rho^2}U(\rho) - 4E_n\frac{U(\rho)}{\rho^2}l(l+1) + 4E_n\frac{n}{\rho}U(\rho)
\end{aligned}$$

Here, we know that $E = E_n$, so we have $E_n$ cancelling:

$$\begin{aligned}
E_n U(\rho) &= 4E_n\frac{\partial^2}{\partial\rho^2}U(\rho) - 4E_n\frac{U(\rho)}{\rho^2}l(l+1) + 4E_n\frac{n}{\rho}U(\rho) \\
U(\rho) &= 4\frac{\partial^2}{\partial\rho^2}U(\rho) - 4\frac{U(\rho)}{\rho^2}l(l+1) + 4\frac{n}{\rho}U(\rho) \\
0 &= 4\frac{\partial^2}{\partial\rho^2}U(\rho) - 4\frac{U(\rho)}{\rho^2}l(l+1) + 4\frac{n}{\rho}U(\rho) - U(\rho) \\
0 &= \frac{\partial^2}{\partial\rho^2}U(\rho) - \frac{U(\rho)}{\rho^2}l(l+1) + \frac{n}{\rho}U(\rho) - \frac{U(\rho)}{4} \\
0 &= \left(\frac{\partial^2}{\partial\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{n}{\rho} - \frac{1}{4}\right)U(\rho)
\end{aligned}$$

## Part e

Use repeated product rules to show that

$$(fgh)'' = f''gh + fg''h + fgh'' + 2f'g'h + 2f'gh' + 2fg'h'$$

Use this and the substitution $U(\rho) = e^{-p/2}\rho^l L(\rho)$ to obtain Laguerre's differential equation

$$\rho L'' + (2l + 2 - \rho)L' + (n - l - 1)L = 0$$

The solutions are associated Laguerre polynomials $L_{n-l-1}^{(2l+1)}(\rho)$.

Assuming that these are all functions in $x$, we get:

$$\begin{aligned}
\frac{\partial}{\partial x}fgh &= \frac{\partial}{\partial x}f(gh) \\
&= f\frac{\partial}{\partial x}gh + f'gh \\
&= f(gh' + g'h) + f'gh \\
&= fgh' + fg'h + f'gh
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}fgh &= \frac{\partial}{\partial x}(fgh' + fg'h + f'gh) \\
&= \frac{\partial}{\partial x}fgh' + \frac{\partial}{\partial x}fg'h + \frac{\partial}{\partial x}f'gh \\
&= \frac{\partial}{\partial x}f(gh') + \frac{\partial}{\partial x}f(g'h) + \frac{\partial}{\partial x}f'(gh) \\
&= f\frac{\partial}{\partial x}(gh') + f'(gh') + f\frac{\partial}{\partial x}(g'h) + f'(g'h) + f'\frac{\partial}{\partial x}(gh) + f''(gh) \\
&= f(gh'' + g'h') + f'gh' + f(g'h' + g''h) + f'g'h + f'(gh' + g'h) + f''gh \\
&= fgh'' + fg'h' + f'gh' + fg'h' + fg''h + f'g'h + f'gh' + f'g'h + f''gh \\
&= fgh'' + 2fg'h' + 2f'gh' + +fg''h + 2f'g'h + f''gh
\end{aligned}$$

Here, it becomes necessary to change the given function, $U(\rho)$ by multiplying through by a factor $\rho$, so we get $\rho U(\rho) = e^{-p/2}\rho^{l+1}L(\rho)$. Doing this, we get:

$$\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial\rho^2} - \frac{l(l+1)}{\rho^2} + \frac{n}{\rho} - \frac{1}{4}\right)\rho U(\rho) \\
0 &= \frac{\partial^2}{\partial\rho^2}\rho U(\rho) - \frac{l(l+1)}{\rho^2}\rho U(\rho) + \frac{n}{\rho}\rho U(\rho) - \frac{1}{4}\rho U(\rho) \\
0 &= \frac{\partial^2}{\partial\rho^2}\rho U(\rho) - \frac{l(l+1)}{\rho^2}e^{-\rho/2}\rho^{l+1}L(\rho) + \frac{n}{\rho}e^{-\rho/2}\rho^{l+1}L(\rho) - \frac{1}{4}e^{-\rho/2}\rho^{l+1}L(\rho) \\
0 &= \frac{\partial^2}{\partial\rho^2}\rho U(\rho) - \frac{l(l+1)}{\rho^2}e^{-\rho/2}\rho^{l+1}L(\rho) + \frac{n}{\rho}e^{-\rho/2}\rho^{l+1}L(\rho) - \frac{1}{4}e^{-\rho/2}\rho^{l+1}L(\rho)
\end{aligned}$$

Now, using the triple product rule from above, we can calculate the second derivative. But first, we find $f'$, $g'$, $h'$, $f''$, $g''$, $h''$, with $f(\rho) = e^{-\rho/2}$, $g(\rho) = \rho^{l+1}$, $h(\rho) = L(\rho)$:

$$\begin{gathered}
f' = -\frac{1}{2}e^{-\rho/2} \\
g' = (l+1)\rho^l \\
h' = L'(\rho) \\
f'' = \frac{1}{4}e^{-\rho/2} \\
g'' = l(l+1)\rho^{l-1} \\
h'' = L''(\rho)
\end{gathered}$$

So, substituting these into the second derivative formula, we get:

$$\begin{aligned}
(fgh)'' = {} & \frac{1}{4}e^{-\rho/2}\rho^{l+1}L(\rho) + e^{-\rho/2}l(l+1)\rho^{l-1}L(\rho) + e^{-\rho/2}\rho^{l+1}L''(\rho) \\
& -2\frac{1}{2}e^{-\rho/2}(l+1)\rho^l L(\rho) - 2\frac{1}{2}e^{-\rho/2}\rho^{l+1}L'(\rho) + 2e^{-\rho/2}(l+1)\rho^l L'(\rho)
\end{aligned}$$

Then, substituting this into the results from part (d), we get:

$$\begin{aligned}
0 = {} & \frac{1}{4}e^{-\rho/2}\rho^{l+1}L(\rho) + e^{-\rho/2}l(l+1)\rho^{l-1}L(\rho) + e^{-\rho/2}\rho^{l+1}L''(\rho) \\
& -2\frac{1}{2}e^{-\rho/2}(l+1)\rho^l L(\rho) - 2\frac{1}{2}e^{-\rho/2}\rho^{l+1}L'(\rho) + 2e^{-\rho/2}(l+1)\rho^l L'(\rho) \\
& -\frac{l(l+1)}{\rho^2}e^{-\rho/2}\rho^{l+1}L(\rho) + \frac{n}{\rho}e^{-\rho/2}\rho^{l+1}L(\rho) - \frac{1}{4}e^{-\rho/2}\rho^{l+1}L(\rho) \\
= {} & \frac{1}{4}\rho^{l+1}L(\rho) + l(l+1)\rho^{l-1}L(\rho) + \rho^{l+1}L''(\rho) - (l+1)\rho^l L(\rho) - \rho^{l+1}L'(\rho) + 2(l+1)\rho^l L'(\rho) \\
& -\frac{l(l+1)}{\rho^2}\rho^{l+1}L(\rho) + \frac{n}{\rho}\rho^{l+1}L(\rho) - \frac{1}{4}\rho^{l+1}L(\rho) \\
= {} & \frac{1}{4}\rho^1 L(\rho) + l(l+1)\rho^{-1}L(\rho) + \rho^1 L''(\rho) - (l+1)L(\rho) - \rho^1 L'(\rho) + 2(l+1)L'(\rho) \\
& -\frac{l(l+1)}{\rho^2}\rho^1 L(\rho) + \frac{n}{\rho}\rho^1 L(\rho) - \frac{1}{4}\rho^1 L(\rho) \\
= {} & l(l+1)\rho^{-1}L(\rho) + \rho L''(\rho) - (l+1)L(\rho) - \rho L'(\rho) + 2(l+1)L'(\rho) \\
& -l(l+1)\rho^{-1}L(\rho) + \frac{n}{\rho}\rho L(\rho) \\
= {} & \rho L''(\rho) - (l+1)L(\rho) - \rho L'(\rho) + 2(l+1)L'(\rho) + nL(\rho) \\
= {} & \rho L''(\rho) + 2(l+1)L'(\rho) - \rho L'(\rho) + nL(\rho) - (l+1)L(\rho) \\
= {} & \rho L''(\rho) + (2l+2-\rho)L'(\rho) + (n-l+1)L(\rho)
\end{aligned}$$

$$\rho L'' + (2l + 2 - \rho) L' + (n - l - 1) L = 0$$

## Part f

Using the values of $L_\kappa^{(\alpha)}$ from `http://mathworld.wolfram.com/LaguerrePolynomial.html`, Eqs. 32-35, substitute back to find the radial wave functions $R_{nl}(r)$ for $n = 1$, 2, and 3. Compare your answers with table 7-2 in the text. What is the physical significance of $\kappa$?

Equations 32 through 35 $L_\kappa^{(\alpha)}$ are:

$$\begin{aligned} L_0^{(\alpha)}(x) &= 1 \\ L_1^{(\alpha)}(x) &= -x + \alpha + 1 \\ L_2^{(\alpha)}(x) &= \frac{1}{2}\left[x^2 - 2(\alpha+2)x + (\alpha+1)(\alpha+2)\right] \end{aligned}$$

Now we can substitute these back into $U_\kappa(\rho) = e^{-p/2}\rho^l L_\kappa^{(\alpha)}(\rho)$, using $\rho = \frac{2r}{r_n}$ and $r_n = \frac{na_0}{Z} = na_0$:

$$\begin{aligned} U_0(\rho) &= e^{-p/2}\rho^l(1) \\ U_1(\rho) &= e^{-p/2}\rho^l(-\rho + \alpha + 1) \\ U_2(\rho) &= e^{-p/2}\rho^l\left(\frac{1}{2}\left[\rho^2 - 2(\alpha+2)\rho + (\alpha+1)(\alpha+2)\right]\right) \end{aligned}$$

$$\begin{aligned} U_0(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}}\left(\frac{2r}{na_0}\right)^l \\ U_1(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}}\left(\frac{2r}{na_0}\right)^l\left(-\frac{2r}{na_0} + \alpha + 1\right) \\ U_2(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}}\left(\frac{2r}{na_0}\right)^l\left(\frac{1}{2}\left[\left(\frac{2r}{na_0}\right)^2 - 2(\alpha+2)\left(\frac{2r}{na_0}\right) + (\alpha+1)(\alpha+2)\right]\right) \end{aligned}$$

For $n = 1$, $l = 0$ (with $\alpha = 2l + 1 = 1$, and $\kappa = n - l - 1 = 0$), we get:

$$\begin{aligned} U_0(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}}\left(\frac{2r}{na_0}\right)^l \\ U_0(\frac{2r}{a_0}) &= e^{-\frac{2r}{2a_0}} \\ U_0(\frac{2r}{a_0}) &= e^{-\frac{r}{a_0}} \end{aligned}$$

For $n = 2$, $l = 0$ (with $\alpha = 2l + 1 = 1$, and $\kappa = n - l - 1 = 1$), we get:

$$\begin{aligned} U_1(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}}\left(\frac{2r}{na_0}\right)^l\left(-\frac{2r}{na_0} + \alpha + 1\right) \\ U_1(\frac{r}{a_0}) &= e^{-\frac{2r}{4a_0}}\left(-\frac{2r}{2a_0} + 1 + 1\right) \\ U_1(\frac{r}{a_0}) &= e^{-\frac{r}{2a_0}}\left(2 - \frac{r}{a_0}\right) \\ U_1(\frac{r}{a_0}) &= 2e^{-\frac{r}{2a_0}}\left(1 - \frac{r}{2a_0}\right) \end{aligned}$$

For $n = 3$, $l = 0$ (with $\alpha = 2l + 1 = 1$, and $\kappa = n - l - 1 = 2$), we get:

$$\begin{aligned}
U_2(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}} \left(\frac{2r}{na_0}\right)^l \left(\frac{1}{2}\left[\left(\frac{2r}{na_0}\right)^2 - 2\left(\alpha+2\right)\left(\frac{2r}{na_0}\right) + \left(\alpha+1\right)\left(\alpha+2\right)\right]\right) \\
U_2(\frac{2r}{3a_0}) &= e^{-\frac{2r}{6a_0}} \left(\frac{1}{2}\left[\left(\frac{2r}{3a_0}\right)^2 - 2\left(1+2\right)\left(\frac{2r}{3a_0}\right) + \left(1+1\right)\left(1+2\right)\right]\right) \\
U_2(\frac{2r}{3a_0}) &= e^{-\frac{r}{3a_0}} \left(\frac{1}{2}\left[\frac{4r^2}{9a_0^2} - 6\left(\frac{2r}{3a_0}\right) + 6\right]\right) \\
U_2(\frac{2r}{3a_0}) &= 3e^{-\frac{r}{3a_0}} \left(1 - \frac{2r}{3a_0} + \frac{2r^2}{27a_0^2}\right)
\end{aligned}$$

Next, we do $n = 2$, $l = 1$ (with $\alpha = 2l + 1 = 3$, and $\kappa = n - l - 1 = 0$), and we get:

$$\begin{aligned}
U_0(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}} \left(\frac{2r}{na_0}\right)^l \\
U_0(\frac{r}{a_0}) &= e^{-\frac{2r}{2a_0}} \left(\frac{2r}{2a_0}\right) \\
U_0(\frac{r}{a_0}) &= e^{-\frac{r}{a_0}} \frac{r}{a_0}
\end{aligned}$$

Next, we do $n = 3$, $l = 1$ (with $\alpha = 2l + 1 = 3$, and $\kappa = n - l - 1 = 1$), and we get:

$$\begin{aligned}
U_1(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}} \left(\frac{2r}{na_0}\right)^l \left(-\frac{2r}{na_0} + \alpha + 1\right) \\
U_1(\frac{2r}{3a_0}) &= e^{-\frac{2r}{6a_0}} \left(\frac{2r}{3a_0}\right) \left(-\frac{2r}{3a_0} + 3 + 1\right) \\
U_1(\frac{2r}{3a_0}) &= e^{-\frac{r}{3a_0}} \left(\frac{2}{3}\right) \left(\frac{r}{a_0}\right) \left(4 - \frac{2r}{3a_0}\right) \\
U_1(\frac{2r}{3a_0}) &= e^{-\frac{r}{3a_0}} \left(\frac{8}{3}\right) \left(\frac{r}{a_0}\right) \left(1 - \frac{r}{6a_0}\right)
\end{aligned}$$

Finally, we do $n = 3$, $l = 2$ (with $\alpha = 2l + 1 = 5$, and $\kappa = n - l - 1 = 0$), and we get:

$$\begin{aligned}
U_0(\frac{2r}{na_0}) &= e^{-\frac{2r}{2na_0}} \left(\frac{2r}{na_0}\right)^l \\
U_0(\frac{2r}{3a_0}) &= e^{-\frac{2r}{6a_0}} \left(\frac{2r}{3a_0}\right)^2 \\
U_0(\frac{2r}{3a_0}) &= e^{-\frac{r}{3a_0}} \left(\frac{4}{9}\right) \left(\frac{r^2}{a_0^2}\right)
\end{aligned}$$

After looking at many pictures, it appears as though $\kappa$ corresponds to the number of radial lobes (plus one) in the probability distribution for the electron orbitals.

## Part g

Show that the ground-state wave function is normalized:

$$\int d^3r \left|\psi_{100}(r,\theta,\phi)\right|^2 = 1$$

We know that $\psi_{100}(r,\theta,\phi) = C_{nlm} R_{21}(r) Y_{00}(\theta,\phi)$ [Tipler & Llewellyn, p. 7-30]. We know that $R_{21}(r) = \frac{2}{\sqrt{a_0^3}} e^{-r/a_0}$ and $Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$Using the integral above, and substituting the spherical Jacobian for $d^3r$, we get:

$$\begin{aligned}
\int d^3r \left|\psi_{100}(r,\theta,\phi)\right|^2 &= 1 \\
\int_0^\infty \int_0^\pi \int_0^{2\pi} \left|C_{nlm} R_{21}(r) Y_{00}(\theta,\phi)\right|^2 r^2 \sin\theta \, d\phi d\theta dr &= 1 \\
\int_0^\infty \int_0^\pi \int_0^{2\pi} \left|C_{nlm} \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \sqrt{\frac{1}{4\pi}}\right|^2 r^2 \sin\theta \, d\phi d\theta dr &= 1 \\
\int_0^\infty \left|C_{nlm} \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \sqrt{\frac{1}{4\pi}}\right|^2 r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi &= 1 \\
2\pi \int_0^\infty \left|C_{nlm} \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \sqrt{\frac{1}{4\pi}}\right|^2 r^2 dr \int_0^\pi \sin\theta \, d\theta &= 1 \\
2\pi \left(-\cos\theta\Big|_0^\pi\right) \int_0^\infty \left|C_{nlm} \frac{2}{\sqrt{a_0^3}} e^{-r/a_0} \sqrt{\frac{1}{4\pi}}\right|^2 r^2 dr &= 1 \\
4\pi \int_0^\infty \frac{1}{4\pi} \frac{4}{a_0^3} C_{nlm}^2 e^{-2r/a_0} r^2 dr &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 \int_0^\infty e^{-2r/a_0} r^2 dr &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 \left(-\frac{a_0}{2} e^{-2r/a_0} r^2 \Big|_0^\infty + a_0 \int_0^\infty e^{-2r/a_0} r dr\right) &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 \left(-\frac{a_0}{2} e^{-2r/a_0} r^2 \Big|_0^\infty + a_0 \left(-\frac{a_0}{2} e^{-2r/a_0} r \Big|_0^\infty + \frac{a_0}{2} \int_0^\infty e^{-2r/a_0} dr\right)\right) &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 \left(-\frac{a_0}{2} e^{-2r/a_0} r^2 + a_0 \left(-\frac{a_0}{2} e^{-2r/a_0} r - \frac{a_0^2}{4} e^{-2r/a_0}\right)\right)\Big|_0^\infty &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 e^{-2r/a_0} \left(-\frac{a_0}{2} r^2 + a_0 \left(-\frac{a_0}{2} r - \frac{a_0^2}{4}\right)\right)\Big|_0^\infty &= 1 \\
\frac{4}{a_0^3} C_{nlm}^2 e^{-2r/a_0} \left(-\frac{a_0}{2} r^2 - \frac{a_0^2}{2} r - \frac{a_0^3}{4}\right)\Big|_0^\infty &= 1 \\
C_{nlm}^2 e^{-2r/a_0} \left(-\frac{2}{a_0^2} r^2 - \frac{2}{a_0} r - 1\right)\Big|_0^\infty &= 1 \\
e^{-2r/a_0} \left(-\frac{2}{a_0^2} r^2 - \frac{2}{a_0} r - 1\right)\Big|_0^\infty &= \frac{1}{C_{nlm}^2} \\
e^{-2\infty/a_0} \left(-\frac{2}{a_0^2}\infty^2 - \frac{2}{a_0}\infty - 1\right) - e^{-20/a_0} \left(-\frac{2}{a_0^2}0^2 - \frac{2}{a_0}0 - 1\right) &= \frac{1}{C_{nlm}^2} \\
e^{-2\infty/a_0} \left(-\frac{2}{a_0^2}\infty^2 - \frac{2}{a_0}\infty - 1\right) + 1 &= \frac{1}{C_{nlm}^2} \\
e^{-2\infty/a_0} \left(-\frac{2}{a_0^2}\infty^2 - \frac{2}{a_0}\infty - 1\right) &= \frac{1}{C_{nlm}^2} - 1
\end{aligned}$$

But, by L'Hopital's rule, we know that $e^{-2\infty/a_0}\left(-\frac{2}{a_0^2}\infty^2 - \frac{2}{a_0}\infty - 1\right) = 0$, so we have that:

$$0 = \frac{1}{C_{nlm}^2} - 1$$

But, this is only true if $C_{nlm}^2 = 1$, which tells us that $\psi(r,\theta,\phi)$ was already normalized.

## Problem 7.26

Show that an electron in the $n = 2$, $l = 1$ state of hydrogen is most likely to be found at $r = 4a_0$.

Generally, $\psi_{nlm}$ is defined by $\psi_{nlm}(r,\theta,\phi) = C_{nlm}R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi) = C_{nlm}R_{nl}(r)Y_{lm}(\theta,\phi)$. We also know that:

$$\begin{aligned} \int \psi^*\psi d\tau &= 1 \\ \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi^*\psi r^2 \sin\theta \, d\phi d\theta dr &= 1 \end{aligned}$$

And, we know that

$$\begin{aligned} R_{21}(r) &= \frac{1}{2\sqrt{6a_0^3}}\frac{r}{a_0}e^{-r/2a_o} \\ Y_{10}(\theta,\phi) &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1\pm1}(\theta,\phi) &= \pm\sqrt{\frac{3}{8\pi}}\sin\theta \, e^{\pm i\phi} \end{aligned}$$

If we recognize that $R_{21}(r)$ does not depend at all upon either $\theta$ or $\phi$, and that neither $Y_{10}(\theta,\phi)$ nor $Y_{1\pm1}(\theta,\phi)$ depend on $r$, we can rewrite the integral from above:

$$\begin{aligned} \int_0^\infty \int_0^\pi \int_0^{2\pi} \psi^*\psi r^2 \sin\theta \, d\phi d\theta dr &= 1 \\ \int_0^\infty \int_0^\pi \int_0^{2\pi} C_{21m}^* R_{21}^* Y_{1m}^* C_{nlm} R_{21} Y_{1m} r^2 \sin\theta \, d\phi d\theta dr &= 1 \\ \int_0^\infty \int_0^\pi \int_0^{2\pi} C_{21m}^* C_{21m} R_{21}^2 Y_{1m}^* Y_{1m} r^2 \sin\theta \, d\phi d\theta dr &= 1 \\ C_{21m}^* C_{21m} \int_0^\infty R_{21}^2 r^2 dr \int_0^\pi \int_0^{2\pi} Y_{1m}^* Y_{1m} \sin\theta \, d\phi d\theta &= 1 \end{aligned}$$

Strictly speaking, the next two steps are not required. $Y(\theta,\phi)$ will only scale the value of the function at $r = 4a_0$, but it will not change that $r = 4a_0$ is where the maximal occurs. However, since it is instructive to see these integrals, and since the work has already been done, they are included. We can now find $\int_0^\pi \int_0^{2\pi} Y_{1m}^* Y_{1m} \sin\theta \, d\phi d\theta$ for both $m = 0$ and $m = \pm 1$. For $m = 0$, we have:

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} Y_{10}^* Y_{10} \sin\theta \, d\phi d\theta &= \int_0^\pi \int_0^{2\pi} Y_{10}^2 \sin\theta \, d\phi d\theta \\ &= \int_0^\pi \int_0^{2\pi} \left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)^2 \sin\theta \, d\phi d\theta \\ &= \int_0^\pi \int_0^{2\pi} \frac{3}{4\pi}\cos^2\theta \sin\theta \, d\phi d\theta \\ &= \frac{3}{4\pi}\int_0^\pi \int_0^{2\pi} \cos^2\theta \sin\theta \, d\phi d\theta \\ &= \frac{3}{4\pi}\int_0^\pi \cos^2\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi \end{aligned}$$

$$\begin{aligned}
&= \frac{3}{4\pi}(2\pi)\int_0^\pi \cos^2\theta\sin\theta\, d\theta \\
&= \frac{3}{2}\int_0^\pi \cos^2\theta\sin\theta\, d\theta \\
&= \frac{3}{2}\left(-\frac{1}{3}\cos^3\theta\Big|_{\theta=0}^{\pi}\right) \\
&= \frac{1}{2}\left(-\left(-1-1\right)\right) \\
&= 1
\end{aligned}$$

For $m = \pm 1$ we have:

$$\begin{aligned}
\int_0^\pi\int_0^{2\pi} Y_{1\pm1}^* Y_{1\pm1}\sin\theta\, d\phi d\theta &= \int_0^\pi\int_0^{2\pi}\left(\pm\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}\right)\left(\pm\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}\right)\sin\theta\, d\phi d\theta \\
&= \int_0^\pi\int_0^{2\pi}\left(\pm\sqrt{\frac{3}{8\pi}}\sin\theta\right)^2\left(e^{-i\phi}\right)\left(e^{i\phi}\right)\sin\theta\, d\phi d\theta \\
&= \int_0^\pi\int_0^{2\pi}\frac{3}{8\pi}\sin^3\theta\left(e^{-i\phi+i\phi}\right)\, d\phi d\theta \\
&= \frac{3}{8\pi}\int_0^\pi\int_0^{2\pi}\sin^3\theta\, d\phi d\theta \\
&= \frac{3}{8\pi}\int_0^\pi\sin^3\theta\, d\theta\int_0^{2\pi}d\phi \\
&= \frac{3}{8\pi}(2\pi)\int_0^\pi\sin^3\theta\, d\theta \\
&= \frac{3}{4}\int_0^\pi\left(1-\cos^2\theta\right)\sin\theta\, d\theta \\
&= \frac{3}{4}\left(\int_0^\pi\sin\theta\, d\theta - \int_0^\pi\cos^2\theta\sin\theta\, d\theta\right) \\
&= \frac{3}{4}\left(-\cos\theta-\left(-\frac{1}{3}\cos^3\theta\right)\right)\Big|_{\theta=0}^{\pi} \\
&= \frac{1}{4}\left(\cos^3\theta - 3\cos\theta\right)\Big|_{\theta=0}^{\pi} \\
&= \frac{1}{4}\left(\left(\cos^3\pi - 3\cos\pi\right)-\left(\cos^3 0 - 3\cos 0\right)\right) \\
&= \frac{1}{4}\left(\cos^3\pi - 3\cos\pi - \cos^3 0 + 3\cos 0\right) \\
&= \frac{1}{4}\left(-1-3(-1)-1+3(1)\right) \\
&= \frac{1}{4}\left(-1+3-1+3\right) \\
&= 1
\end{aligned}$$

So, we now know that $Y_{1m}(\theta,\phi) = 1$. So we're left with:

$$C_{21m}^* C_{21m}\int_0^\infty R_{21}^2 r^2 dr = 1$$

However, looking ahead, we know that we will be finding where the derivative of some function is zero, and a constant factor will not change where the derivative is zero, just the magnitude of the function at that point. So, we have $\int_0^\infty C_{21m}^* C_{21m} R_{21}^2 r^2 dr = 1$, which looks like a probability distribution integration, of the form $\int_0^\infty P(r)\,dr = 1$. So, let's let $P(r)\,dr = C^*_{21m}C_{21m}R^2_{21}r^2dr$, so $P(r) = C^*_{21m}C_{21m}R^2_{21}r^2$, and the maximum probability will occur at $\frac{dP}{dr} = 0$. This is:

$$\begin{aligned}
\frac{d}{dr}P(r) &= 0 \\
\frac{d}{dr}\left(C^*_{21m}C_{21m}R^2_{21}r^2\right) &= 0 \\
C^*_{21m}C_{21m}\frac{d}{dr}\left(\left(\frac{1}{2\sqrt{6a_0^3}}\frac{r}{a_0}e^{-r/2a_o}\right)^2 r^2\right) &= 0 \\
C^*_{21m}C_{21m}\frac{d}{dr}\left(\left(\frac{1}{2a_0\sqrt{6a_0^3}}\right)^2 r^2e^{-r/a_o}r^2\right) &= 0 \\
C^*_{21m}C_{21m}\left(\frac{1}{2a_0\sqrt{6a_0^3}}\right)^2\frac{d}{dr}\left(r^4e^{-r/a_o}\right) &= 0 \\
C^*_{21m}C_{21m}\left(\frac{1}{2a_0\sqrt{6a_0^3}}\right)^2\left(r^4\left(-\frac{1}{a_0}\right)e^{-r/a_0} + 4r^3e^{-r/a_0}\right) &= 0 \\
C^*_{21m}C_{21m}\left(\frac{1}{2a_0\sqrt{6a_0^3}}\right)^2 r^3e^{-r/a_0}\left(-\frac{r}{a_0}+4\right) &= 0
\end{aligned}$$

Since $e^{-r/a_0}$ is never zero, and since $r = 0$ is impossible because $l = 1$ so the electron must have angular momentum which would not happen at $r = 0$, we have:

$$\begin{aligned}
-\frac{r}{a_0}+4 &= 0 \\
-\frac{r}{a_0} &= -4 \\
r &= -4(-a_0) \\
r &= 4a_0
\end{aligned}$$

# Problem 7.29

If a classical system does not have a constant charge-to-mass ratio throughout the system, the magnetic moment can be written

$$\mu = g\frac{Q}{2M}L$$

where $Q$ is the total charge, $M$ is the total mass, and $g \neq 1$.

## Part a

Show that $g = 2$ for a solid cylinder ($I = \frac{1}{2}MR^2$) that spins about its axis and has a uniform charge on its cylindrical surface.

We know that $\mu = iA$. We also know, in this case, that the area of the loop about which the charge is circulating is $A = \pi R^2$. To find $i$, we must recognize that current is equal to charge times the frequency, or $i = Qf$, that $f = \frac{\omega}{2\pi}$, and that $L = I\omega$. This gives us:

$$\mu = iA$$

$$
\begin{aligned}
&= Qf\pi R^2 \\
&= Q\frac{\omega}{2\pi}\pi R^2 \\
&= Q\frac{1}{2}\left(\frac{L}{I}\right)R^2 \\
&= Q\frac{1}{2}\left(\frac{L}{\frac{1}{2}MR^2}\right)R^2 \\
&= Q\left(\frac{L}{M}\right) \\
&= 2\frac{Q}{2M}L
\end{aligned}
$$

So, $g = 2$.

### Part b

> Show that $g = 2.5$ for a solid sphere ($I = 2MR^2/5$) that has a ring of charge on the surface at the equator, as shown in Figure 7-33 [Tipler & Llewellyn, p. 309].

In this case, practically everything is identical to part (a) except for the moment of inertia. So we have:

$$
\begin{aligned}
\mu &= iA \\
&= Qf\pi R^2 \\
&= Q\frac{\omega}{2\pi}\pi R^2 \\
&= Q\frac{1}{2}\left(\frac{L}{I}\right)R^2 \\
&= Q\frac{1}{2}\left(\frac{L}{\frac{2}{5}MR^2}\right)R^2 \\
&= \frac{5}{4}Q\left(\frac{L}{M}\right) \\
&= \frac{5}{2}\cdot\frac{Q}{2M}L
\end{aligned}
$$

So, $g = \frac{5}{2} = 2.5$.

## Problem 7.39

> Consider a system of two electrons, each with $l = 1$ and $s = \frac{1}{2}$.

### Part a

> What are the possible values of the quantum number for the total orbital angular momentum $\vec{L} = \vec{L}_1 + \vec{L}_2$?

The quantum number, $L$, for $\vec{L}$ has possible values $l_1 + l_2, l_1 + l_2 - 1, \ldots, |l_1 - l_2|$, where $l_1$ and $l_2$ are the total orbital angular momentum quantum numbers for $\vec{L}_1$ and $\vec{L}_2$ respectively. These are both 1, so, we have that $L = 2$, $L = 1$, or $L = 0$.

### Part b

What are the possible values of the quantum number $S$ for the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$?

Similarly to the above, we have quantum numbers, $s_1$ and $s_2$ equal to $\frac{1}{2}$, so we have the total spin quantum number as $S = 1$ or $S = 0$.

### Part c

Using the results of parts (a) and (b), find the possible quantum numbers $j$ for the combination $\vec{J} = \vec{L} + \vec{S}$.

The possible quantum numbers for $j$ can be either $j = L + S$ or $j = |L - S|$. Since we have multiple possibilities for $L$ and $S$, we try each combination to find all possible quantum numbers. So, for $j$ we have: $2 + 1 = 3$, $2 - 1 = 1$, $2 + 0 = 2$, $2 - 0 = 2$, $1 + 1 = 2$, $1 - 1 = 0$, $1 + 0 = 1$, $1 - 0 = 0$, $0 + 1 = 1$, $|0 - 1| = 1$, $0 + 0 = 0$, $0 - 0 = 0$. So, $j$ can equal: 3, 2, 1, or 0.

### Part d

What are the possible quantum numbers $j_1$ and $j_2$ for the total angular momentum of each particle?

Since $l_1 = l_2 = 1$ and $s_1 = s_2 = \frac{1}{2}$, both $j_1$ and $j_2$ have the same possible quantum numbers. These are given, as in part (c), by $j_1 = l_1 + s_1$ or $j_1 = |l_1 - s_1|$. So, we get the possible quantum numbers for $j_1 = j_2$ to be 1.5 or 0.5.

### Part e

Use the results of part (d) to calculate the possible values of $j$ from the combinations of $j_1$ and $j_2$. Are these the same as in part (c)?

We know that that the quantum number, $j$, for $\vec{J}$ has possible values $j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|$, So, using the results from part (d), we see that the possible values for $j$ are the integers between $1.5 + 1.5 = 3$ and $1.5 - 1.5 = 0$, or between $1.5 + 0.5 = 2$ and $1.5 - 0.5 = 1$, or between $0.5 + 1.5 = 2$ and $|0.5 - 1.5| = 1$, or between $0.5 + 0.5 = 1$ and $0.5 - 0.5 = 0$. So, all the possible values are: 3, 2, 1, and 0. This is the same as in part (c).

## Problem 7.44

Write the electron configuration of the following elements:

### Part a

Carbon

Carbon has $Z = 6$, so its electron configuration is $1s^2 2s^2 2p^2$.

### Part b

Oxygen

Oxygen has $Z = 8$, so its electron configuration is $1s^2 2s^2 2p^4$.

### Part c

Argon

Argon has $Z = 18$, so its electron configuration is $1s^2 2s^2 2p^6 3s^2 3p^6$.

## Problem 7.73

In the anomalous Zeeman effect, the external magnetic field is much weaker than the internal field seen by the electron as a result of its orbital motion. In the vector model (Figure 7-30 [www.whfreeman.com/tiplermodernphysics5e]) the vectors $\vec{L}$ and $\vec{S}$ precess rapidly around $\vec{J}$ because of the internal field and $\vec{J}$ precesses slowly around the external field. The energy splitting is found by first calculating the component of the magnetic moment $\mu_J$ in the direction of $\vec{J}$ and then finding the component of $\vec{\mu}_z$ in the direction of $\vec{B}$.

### Part a

Show that $\mu_J = \frac{\vec{\mu}\cdot\vec{J}}{J}$ can be written

$$\mu_J = -\frac{\mu_B}{\hbar J}\left(L^2 + 2S^2 + 3\vec{S}\cdot\vec{L}\right)$$

We can substitute $\vec{\mu} = \frac{-g_L\mu_B\vec{L}}{\hbar} + \frac{-g_S\mu_B\vec{S}}{\hbar} = -\frac{\mu_B}{\hbar}\left(g_L\vec{L} + g_S\vec{S}\right)$, where $g_L = 1$ and $g_S \approx 2$ ([Tipler & Llewellyn, p. 287]), and $\vec{J} = \vec{L} + \vec{S}$, and we get:

$$\begin{aligned}
\mu_J &= \frac{\vec{\mu}\cdot\vec{J}}{J} \\
&= \frac{\left(-\frac{\mu_B\left(\vec{L}+2\vec{S}\right)}{\hbar}\right)\cdot\left(\vec{L}+\vec{S}\right)}{J} \\
&= \frac{\left(\frac{-\mu_B}{\hbar}\right)\left(\left(\vec{L}+2\vec{S}\right)\cdot\left(\vec{L}+\vec{S}\right)\right)}{J} \\
&= \frac{-\mu_B}{\hbar J}\left(\left(\vec{L}+2\vec{S}\right)\cdot\left(\vec{L}+\vec{S}\right)\right) \\
&= \frac{-\mu_B}{\hbar J}\left(\vec{L}\cdot\left(\vec{L}+\vec{S}\right) + 2\vec{S}\cdot\left(\vec{L}+\vec{S}\right)\right) \\
&= \frac{-\mu_B}{\hbar J}\left(\left(\vec{L}\cdot\vec{L} + \vec{L}\cdot\vec{S}\right) + \left(2\vec{S}\cdot\vec{L} + 2\vec{S}\cdot\vec{S}\right)\right) \\
&= \frac{-\mu_B}{\hbar J}\left(L^2 + \vec{L}\cdot\vec{S} + 2\vec{L}\cdot\vec{S} + 2S^2\right) \\
&= -\frac{\mu_B}{\hbar J}\left(L^2 + 2S^2 + 3\vec{L}\cdot\vec{S}\right)
\end{aligned}$$

### Part b

From $J^2 = \left(\vec{L}+\vec{S}\right)\cdot\left(\vec{L}+\vec{S}\right)$ show that $\vec{S}\cdot\vec{L} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$.

This is, easily:

$$\begin{aligned}
J^2 &= \left(\vec{L}+\vec{S}\right)\cdot\left(\vec{L}+\vec{S}\right) \\
J^2 &= \left(\vec{L}\cdot\left(\vec{L}+\vec{S}\right) + \vec{S}\cdot\left(\vec{L}+\vec{S}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
J^2 &= \vec{L}\cdot\vec{L} + \vec{L}\cdot\vec{S} + \vec{S}\cdot\vec{L} + \vec{S}\cdot\vec{S} \\
J^2 &= \vec{L}\cdot\vec{L} + \vec{S}\cdot\vec{L} + \vec{S}\cdot\vec{L} + \vec{S}\cdot\vec{S} \\
J^2 &= L^2 + 2\vec{S}\cdot\vec{L} + S^2 \\
\frac{1}{2}\left(J^2 - L^2 - S^2\right) &= \vec{S}\cdot\vec{L}
\end{aligned}
$$

## Part c

Substitute your result in part (b) into that of part (a) to obtain

$$\mu_J = -\frac{\mu_B}{2\hbar J}\left(3J^2 + S^2 - L^2\right)$$

This becomes:

$$
\begin{aligned}
\mu_J &= -\frac{\mu_B}{\hbar J}\left(L^2 + 2S^2 + 3\vec{L}\cdot\vec{S}\right) \\
&= -\frac{\mu_B}{\hbar J}\left(L^2 + 2S^2 + 3\frac{1}{2}\left(J^2 - L^2 - S^2\right)\right) \\
&= -\frac{\mu_B}{2\hbar J}\left(2L^2 + 4S^2 + 3J^2 - 3L^2 - 3S^2\right) \\
&= -\frac{\mu_B}{2\hbar J}\left(3J^2 - L^2 + S^2\right)
\end{aligned}
$$

## Part d

Multiply your result by $J_z/J$ to obtain

$$\mu_z = -\mu_B\left(1 + \frac{J^2 + S^2 - L^2}{2J^2}\right)\frac{J_z}{\hbar}$$

This becomes:

$$
\begin{aligned}
-\frac{\mu_B}{2\hbar J}\left(3J^2 - L^2 + S^2\right)\left(\frac{J_z}{J}\right) &= -\frac{\mu_B J_z}{2\hbar J^2}\left(3J^2 - L^2 + S^2\right) \\
&= -\frac{\mu_B J_z}{\hbar}\left(\frac{3J^2 - L^2 + S^2}{2J^2}\right) \\
&= -\mu_B\left(\frac{2J^2}{2J^2} + \frac{J^2 - L^2 + S^2}{2J^2}\right)\frac{J_z}{\hbar} \\
&= -\mu_B\left(1 + \frac{J^2 - L^2 + S^2}{2J^2}\right)\frac{J_z}{\hbar}
\end{aligned}
$$