

Problem 1

[30 pts] While the spins of the two electrons in a hydrogen molecule must be antiparallel in the ground state, there is a degeneracy due to the spin of the two protons, $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$.

Part a

Why doesn't the Pauli exclusion principle apply to the protons? In other words, why can the two protons have arbitrary m_s , while the electrons must occupy different states?

The Pauli exclusion principle does not apply to the protons in a Hydrogen molecule because the protons are not shared by the atoms in the molecule, while the electrons are shared and inhabit the same molecular shell in the ground state (and so must have different spins).

Part b

List the possible combinations of quantum numbers (m_{s1}, m_{s2}) . What is the degeneracy of the ground state?

Since protons have spin $\frac{1}{2}$, we have $m_s = \pm\frac{1}{2}$. So, the possible combinations of quantum numbers are: $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$. The degeneracy of the ground state is 4.

Part c

Show how the proton spins s_1 and s_2 couple to form a singlet and triplet. List the possible quantum numbers (s, m_s) of the total spin of the protons $\vec{s} = \vec{s}_1 + \vec{s}_2$, and find the degeneracy g_s for each value of the total spin s . This explains the names "singlet" and "triplet."

So, the possible proton spins couple to give $s = s_1 + s_2, s_1 + s_2 - 1, \dots, |s_1 - s_2|$, and have possible values for $m_s = m_{s1} + m_{s2}$ for each combination of m_{s1} and m_{s2} . So the s couplings are $s = \frac{1}{2} + \frac{1}{2}, |\frac{1}{2} - \frac{1}{2}| = 1, 0$, and $m_s = \frac{1}{2} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}, -\frac{1}{2} - \frac{1}{2} = 1, 0, 0, -1$. The possible quantum number pairs, (s, m_s) are $(1, 1), (1, 0), (1, -1)$, and $(0, 0)$. This is because you can't have $s = 0$ with an m_s value of anything but 0, but it is possible to find a vector \vec{s} with $s = 1$, but $m_s = 0$. So, we have a triplet with coupled spins $m_s = -1, 0, 1$ and a singlet with coupled spin $s = 0$. The degeneracy for the triplet is $g_1 = 3$ for the triplet, since it has three coupled spin states, and $g_0 = 1$ for the singlet since it has only one spin state.

Part d

Ortho-hydrogen (spins aligned, triplet) is 15 meV higher in energy than the ground state of para-hydrogen $E = 0$ (spins anti-parallel, singlet). For which value of s is it possible to have $(m_{s1} = m_{s2}) = \frac{1}{2}$ (spin aligned)?

The only value of s for which it is possible to have spin $m_{s1} = m_{s2} = \frac{1}{2}$ is $s = 1$.

Part e

Using g_s and $f_{MB} = e^{-\epsilon/kT}$, calculate and plot the fraction of ortho- and para-hydrogen as a function of temperature from $T = 0$ K to $T = 300$ K.

We know that g_s should be constant for $s = 0$ and $s = 1$, and we have that for para-hydrogen $f_{MB} = A(T) e^{-0/kT} = 1$, and for ortho-hydrogen $f_{MB} = A(T) e^{-15/kT}$. We also know that $n_s(T, E_s) = g_s f_{MB}(T, E_s)$ and that for properly normalized distributions, $n_0(T, E_0) = n_0(T, 0)$ and $n_1(T, E_1) = n_1(T, 15)$, $1 = n_0(T, 0) + n_1(T, 15)$. So we have:

$$\begin{aligned} 1 &= n_0(T, 0) + n_1(T, 15) \\ 1 &= A(T) + 3A(T) e^{-15/kT} \\ 1 &= A(T) (1 + 3e^{-15/kT}) \\ \frac{1}{(1 + 3e^{-15/kT})} &= A(T) \end{aligned}$$

So the fraction of molecules for $s = 0$ and $s = 1$ is $n_0(T) = \frac{1}{(1 + 3e^{-15/kT})}$ and $n_1(T) = 3 \frac{1}{(1 + 3e^{-15/kT})} e^{-15/kT}$. Plotting these, we get figure 1.

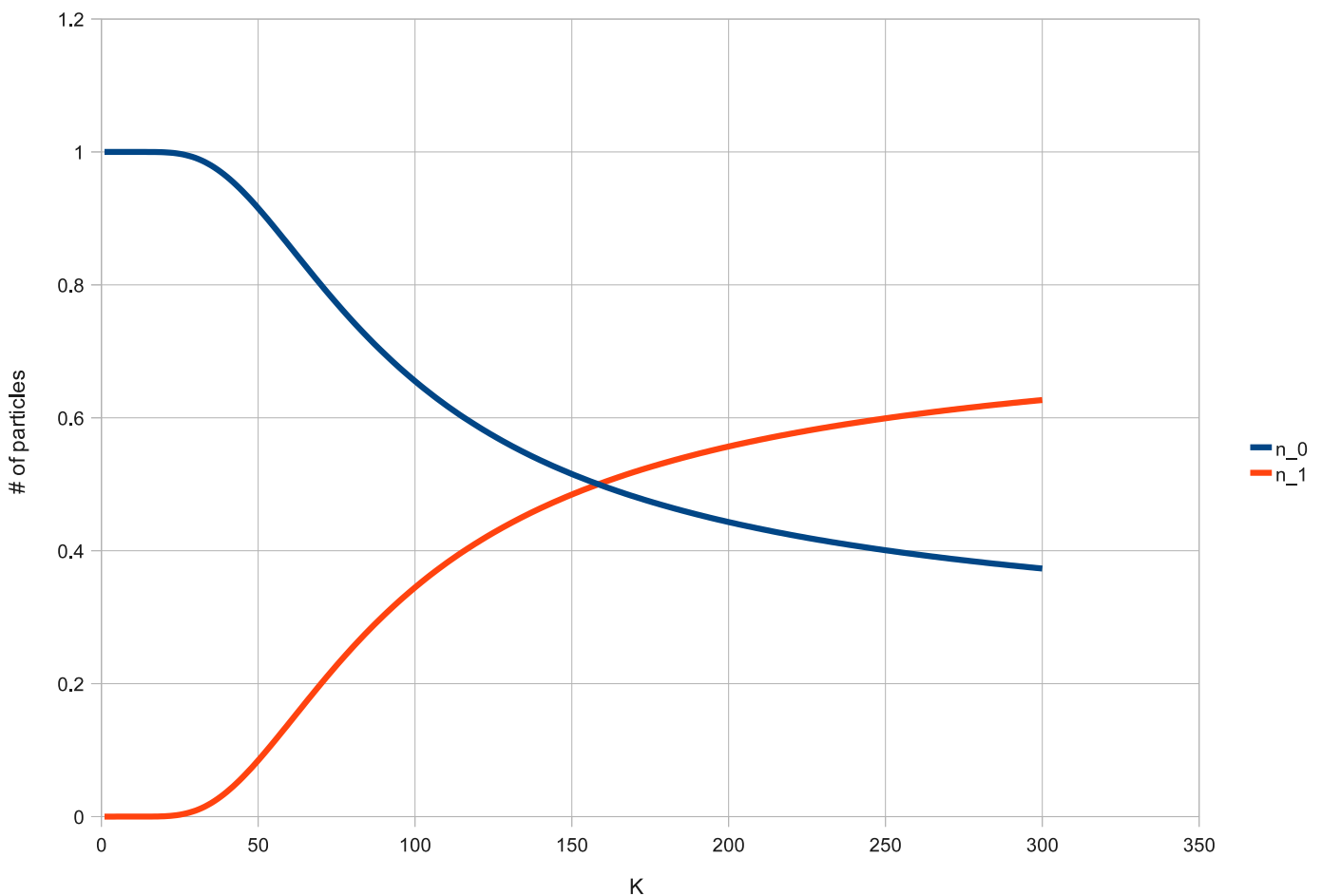


Figure 1: The fraction of para and ortho hydrogen from 0 K to 300 K.

Part f

Calculate and plot the average energy per molecule as a function of temperature.

Since the molecules in the ground state have zero energy, the average energy per molecule is just the contribution to energy from the ortho-hydrogen, 15 meV, times the fraction of the molecules in that state, which is:

$$\begin{aligned}\langle \epsilon \rangle (T) &= 15n_1(T) \\ &= (15)(3) \frac{1}{1 + 3e^{-15/kT}} e^{-15/kT}\end{aligned}$$

Plotting this, we get figure 2.

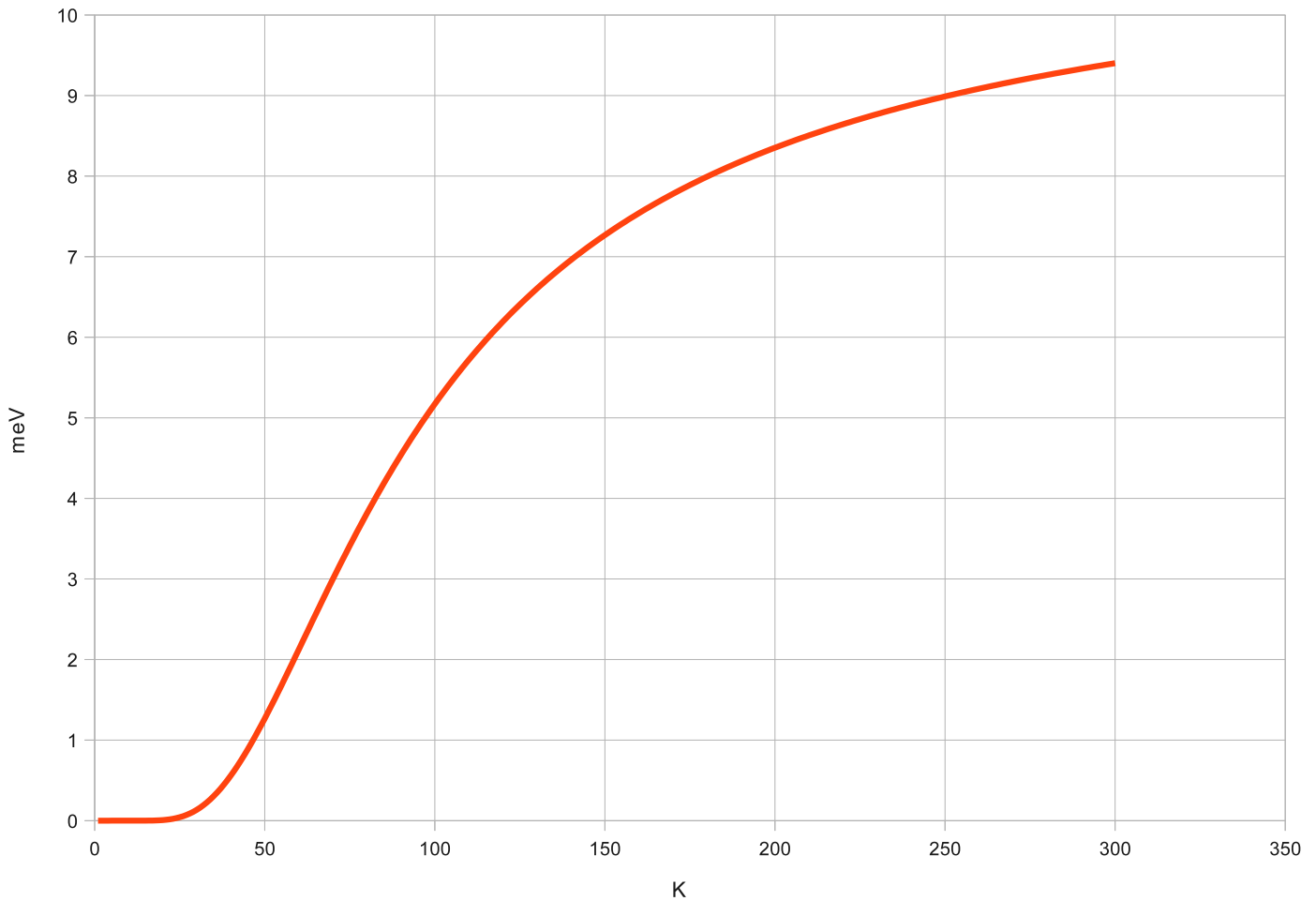


Figure 2: The average energy of hydrogen from 0 K to 300 K.

Problem 2

[30 pts] Helium atoms have total spin $s = 0$, following the Bose-Einstein distribution

$$f_{BE}(\epsilon) = \frac{1}{e^{\alpha} e^{\epsilon/kT} - 1}$$

Part a

Starting with $hdn_x = dp_x dx$ where dn_x is the number of available states in the range dp_x of momentum and dx in space in the x -direction, show that $dn = 4\pi p^2 dp V / h^3$ in three dimensions, where V is the volume and $dn = dn_x dn_y dn_z$.

While it does make sense to say that $hdn_x = dp_x dx$ since we are dealing with infinitesimals, it will be easier to show how these equations balance if we use $hd^2 n_x = dp_x dx$, $d^6 n = d^2 n_x d^2 n_y d^2 n_z$, and $d^6 n = 4\pi p^2 d^3 p V / h^3$. So, if $hd^2 n_x = dp_x dx$, then we have $hd^2 n_y = dp_y dy$ and $hd^2 n_z = dp_z dz$. So we can get $d^6 n = d^2 n_x d^2 n_y d^2 n_z$ by:

$$\begin{aligned} d^6 n &= d^2 n_x d^2 n_y d^2 n_z \\ &= d^2 n_x d^2 n_y d^2 n_z \\ &= hd^2 n_x hd^2 n_y hd^2 n_z \\ &= dp_x dx dp_y dy dp_z dz \\ &= d^3 p d^3 x \end{aligned}$$

But, we know that we can integrate, getting:

$$\begin{aligned} \int \int \int d^6 n &= \int \int \int d^3 p d^3 x \\ d^3 n &= d^3 p V \end{aligned}$$

Also, we have that the density of states for momentum is equal to the number of states in the sphere defined by $p = \frac{4}{3}\pi r^3$, where r is the radius of the space in three components, p_x , p_y and p_z . So, we have $d^3 p = 4\pi r^2 dr$. This gives us:

$$d^3 n = 4\pi r^2 dr V / h^3$$

However, since $p = r = \sqrt{p_x^2 + p_y^2 + p_z^2}$ and $d^3 r = dp_x dp_y dp_z = d^3 p$, we get:

$$d^3 n = 4\pi p^2 d^3 p V / h^3$$

Part b

Calculate the degeneracy $g(\epsilon)$ of kinetic energy states $\epsilon = p^2/2m$, defined by $dn = g(\epsilon) d\epsilon$.

So, we have that $p = \sqrt{\epsilon 2m}$, and $dp = \sqrt{2m} \frac{1}{2} (\epsilon)^{-\frac{1}{2}} d\epsilon$. This gives us

$$\begin{aligned} 4\pi p^2 dp V / h^3 &= g(\epsilon) d\epsilon \\ 4\pi \epsilon m \sqrt{2m} \epsilon^{-\frac{1}{2}} V / h^3 d\epsilon &= g(\epsilon) d\epsilon \end{aligned}$$

$$\text{So, } g(\epsilon) = 4\pi m \sqrt{2m} \epsilon^{\frac{1}{2}} V / h^3$$

Part c

Insert $g(\epsilon)$ and $f(\epsilon)$ into $N = \int_0^\infty d\epsilon g(\epsilon) f_{BE}(\epsilon)$ to get a formula for the N , the total number of atoms. Show that the number density is

$$\frac{N}{V} = \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(\alpha)$$

where

$$I(\alpha) = \int_0^\infty \frac{\sqrt{x} dx}{e^{\alpha+x} - 1}$$

Substituting $g(\epsilon)$ and $f(\epsilon)$ into $N = \int_0^\infty d\epsilon g(\epsilon) f_{BE}(\epsilon)$ gives us:

$$N = \int_0^\infty d\epsilon \frac{4\pi V m \epsilon^{\frac{1}{2}} \sqrt{2m}}{h^3} \frac{1}{e^{\alpha} e^{\epsilon/kT} - 1}$$

For $\frac{N}{V}$, where $x = \frac{\epsilon}{kT}$ (so $\epsilon = xkT$ and $d\epsilon = dxkT$), then, we get:

$$\begin{aligned} \frac{N}{V} &= \int_0^\infty d\epsilon \frac{4\pi m \epsilon^{\frac{1}{2}} \sqrt{2m}}{h^3} \frac{1}{e^{\alpha} e^{\epsilon/kT} - 1} \\ &= \int_0^\infty d\epsilon \frac{4\pi m \sqrt{kTx} \sqrt{2m}}{h^3} \frac{1}{e^{\alpha} e^x - 1} \\ &= \int_0^\infty d\epsilon \frac{4\pi m \sqrt{kTx} \sqrt{2m}}{h^3} \frac{1}{e^{\alpha+x} - 1} \\ &= \int_0^\infty dx kT \frac{4\pi m \sqrt{kTx} \sqrt{2m}}{h^3} \frac{1}{e^{\alpha+x} - 1} \\ &= \int_0^\infty \frac{2\pi (2mkT)^{\frac{3}{2}} \sqrt{x} dx}{h^3} \frac{1}{e^{\alpha+x} - 1} \\ &= \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} \int_0^\infty \frac{\sqrt{x} dx}{e^{\alpha+x} - 1} \\ &= \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(\alpha) \end{aligned}$$

Part d

Plot the integrand of $I(\alpha)$ for $\alpha = 0, 0.5$, and 1 to show that the integral decreases as a function of α . Circle the singularity in the integrand for $\alpha = 0$. Numerically, $I(0) = 2.315$, $I(0.5) = 0.7183$, and $I(1) = 0.3797$.

Plotting these functions, we get the graph in figure 3. The singularity in the integrand occurs at $x = 0$ for $\alpha = 0$.

Part e

In the above formula, α changes as a function of temperature T to keep the number density N/V fixed at the experimental density of liquid helium, $\rho = 0.146 \text{ g/mL}$ at T_λ . Calculate the temperature corresponding to $\alpha = 0, 0.5$, and 1 .

We have

$$\begin{aligned} \frac{N}{V} &= \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(\alpha) \\ 0.146 \text{ g/mL} &= \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(\alpha) \\ \frac{h^3 0.146 \text{ g/mL}}{2\pi (2mk)^{\frac{3}{2}} I(\alpha)} &= T^{\frac{3}{2}} \\ \left(\frac{h^3 0.146 \text{ g/mL}}{2\pi (2mk)^{\frac{3}{2}} I(\alpha)} \right)^{\frac{2}{3}} &= T \\ \frac{h^2 (0.146 \text{ g/mL})^{\frac{2}{3}}}{(2\pi)^{\frac{2}{3}} (2mk)^{\frac{2}{3}} I(\alpha)^{\frac{2}{3}}} &= T \end{aligned}$$

Substituting in for α for $T(\alpha)$, we get $T(0) \approx 3.1 \text{ K}$, $T(0.5) \approx 6.9 \text{ K}$, $T(1) \approx 10.5 \text{ K}$.

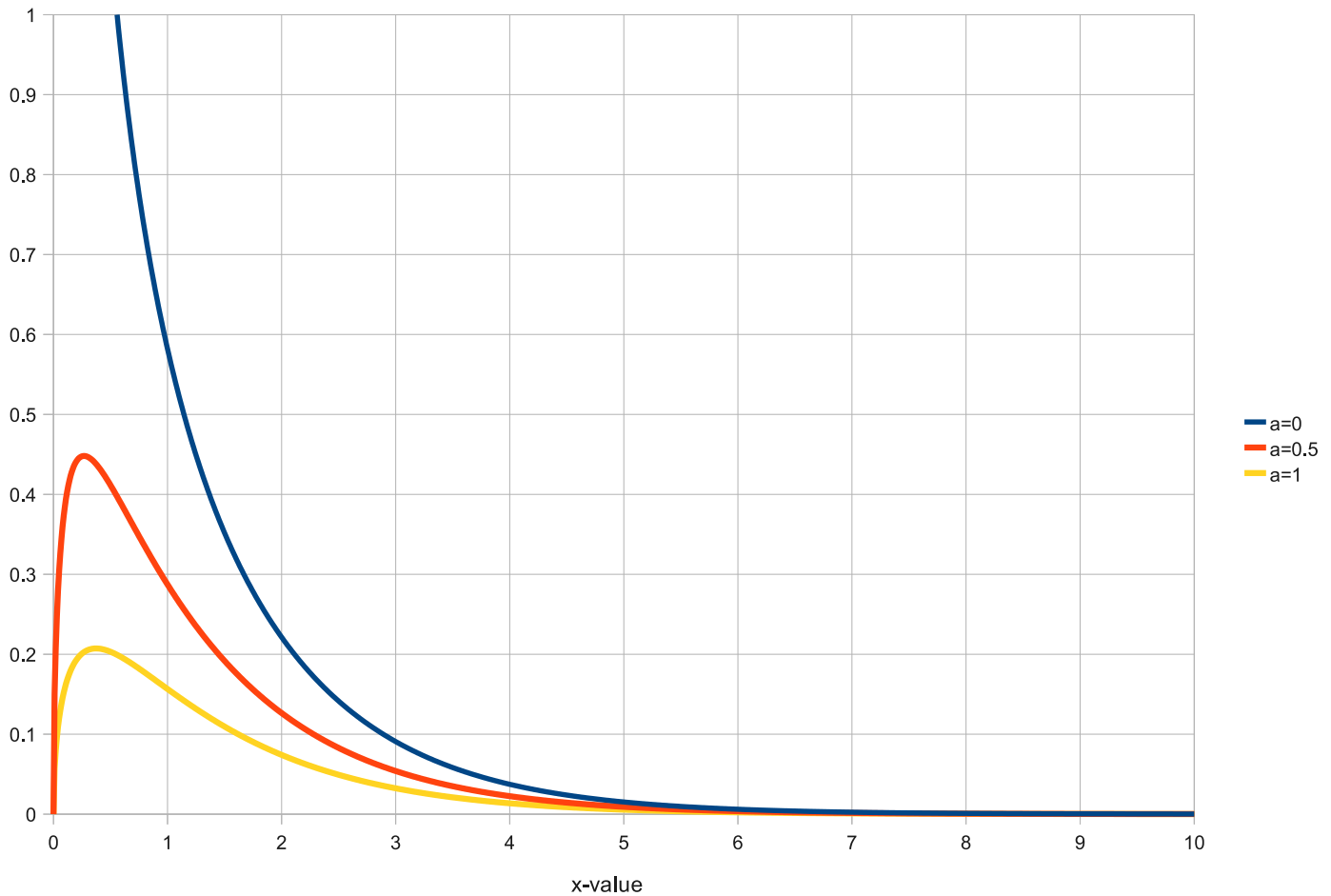


Figure 3: Plot of $\frac{\sqrt{x}}{e^{\alpha x} - 1}$ for $\alpha = 0, 0.5, 1$

Part f

Because of the singularity at $\alpha = 0$, the value of α must remain positive, even as T drops below the critical temperature T_λ (when $\alpha = 0$). Therefore, the integral $I(\alpha)$ is a constant for $T < T_\lambda$, and the density of the “normal” fluid helium drops below the critical temperature. The rest of the atoms condense into the ground state to form a “superfluid” component with zero viscosity. Plot the density of the normal and superfluid components of Helium II as a function of T below the lambda point T_λ , keeping the total density constant. Note: the experimental value for the critical point is $T_\lambda = 2.17$ K.

If the density, $\rho = 0.146$ g/mL remains constant, we have, where $\frac{N_n}{V}$ is the density of the normal helium and $\frac{N_s}{V}$ is the density of the superfluid:

$$\begin{aligned} \frac{N}{V} &= 2.20 \times 10^{28} \text{ particles/m}^3 = \frac{N_n}{V} + \frac{N_s}{V} \\ 2.20 \times 10^{28} \text{ particles/m}^3 - \frac{N_n}{V} &= \frac{N_s}{V} \\ 2.20 \times 10^{28} \text{ particles/m}^3 - \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} I(0) &= \frac{N_s}{V} \end{aligned}$$

$$2.20 \times 10^{28} \text{ particles/m}^3 = \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} (2.315) = \frac{N_s}{V}$$

This yields the graph in figure 4 for $\frac{N_n}{V}$ and $\frac{N_s}{V}$.

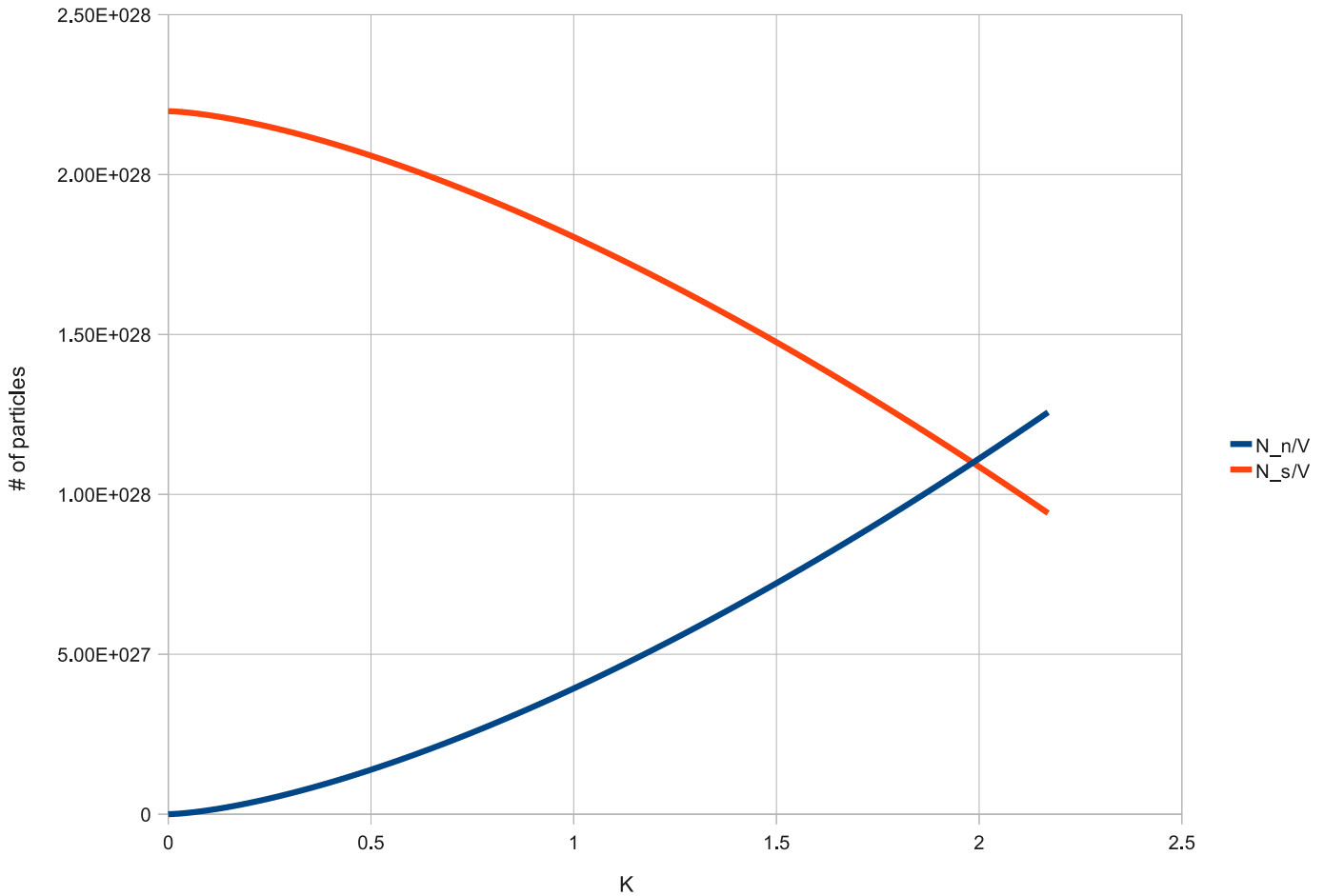


Figure 4: Plot of the density of normal to superfluid hydrogen as a function of temperature

Problem 3

[30 pts] A neutron star occurs when a star of up to 1.5 solar masses collapses under its own weight. Protons in the nuclei decay into neutrons, which have no electrical repulsion, and the only thing which prevents further collapse into a black hole is the Fermi repulsion of the neutrons, which have spin ($s = \frac{1}{2}$). They follow the Fermi-Dirac distribution

$$f_{FD}(\epsilon) = \frac{1}{e^{\alpha} e^{\epsilon/kT} + 1}$$

Part a

Calculate the total number of neutrons created from a star 1.5 times the mass of the sun, $m_{sun} = 1.9891 \times 10^{30}$ kg. (think big)

Here, we just have

$$\begin{aligned}
 N &= \frac{m_{sun}}{m_{neutron}} \\
 &= \frac{1.5 * 1.98892 \times 10^{30} \text{ kg}}{1.67492729 \times 10^{-27} \text{ kg}} \\
 &= 1.78119971 \times 10^{57} \text{ neutrons}
 \end{aligned}$$

Part b

As in problem #2, show that the density of states is $dN = 8\pi p^2 dpV/h^3$. The extra factor of 2 comes from the two spin states of the neutron. In the zero temperature limit, each state is filled with one neutron, and all of the states are filled up to the maximum (Fermi) momentum p_F . Calculate the total number of neutrons $N = \int_0^{p_F} dN$ in a spherical volume (star) of radius R .

Here, we can use $hdN = dpdx$. This gives us that $hd^6N = d^3pd^3x$. Integrating, we get:

$$\begin{aligned}
 \int \int \int d^6N &= \int \int \int d^3pd^3x \\
 d^3N &= d^3pV
 \end{aligned}$$

Also, we have that the density of states for momentum is equal to the number of states in the sphere defined by $p = \frac{4}{3}\pi r^3$, where r is the radius of the space in three components, p_x , p_y and p_z . So, we have $d^3p = 4\pi r^2 dr$. This gives us:

$$d^3nN = 4\pi r^2 drV/h^3$$

However, since $r = \sqrt{p_x^2 + p_y^2 + p_z^2} = p$ and $d^3r = dp_x dp_y dp_z = d^3p$, we get:

$$d^3N = 4\pi p^2 d^3pV/h^3$$

Of course, if each of these momentum and spacial states can have one of two spins, that doubles the number of states, so that we get:

$$d^3N = 8\pi p^2 d^3pV/h^3$$

If we have a spherical volume of radius R , the total number of neutrons, between momentums 0 and p_F , is:

$$\begin{aligned}
 N &= \int_0^{p_F} 8\pi p^2 dpV/h^3 \\
 &= \int_0^{p_F} 8\pi p^2 dp \frac{4}{3}\pi R^3/h^3 \\
 &= \frac{32}{3} \frac{\pi^2 R^3}{h^3} \int_0^{p_F} p^2 dp \\
 &= \frac{32}{3} \frac{\pi^2 R^3}{h^3} \frac{1}{3} (p^3) \Big|_0^{p_F} \\
 &= \frac{32}{9} \frac{\pi^2 R^3 p_F^3}{h^3}
 \end{aligned}$$

Part c

Calculate the weighted average kinetic energy of each neutron as a function of p_F , using

$$\langle E_{kin} \rangle = \frac{\int_0^{p_F} dN \frac{p^2}{2M_n}}{\int_0^{p_F} dN} = \frac{\int_0^{p_F} p^2 dp \frac{p^2}{2M_n}}{\int_0^{p_F} p^2 dp}$$

Starting with $\langle E_{kin} \rangle = \frac{\int_0^{p_F} p^2 dp \frac{p^2}{2M_n}}{\int_0^{p_F} p^2 dp}$, we get:

$$\begin{aligned} \langle E_{kin} \rangle &= \frac{\int_0^{p_F} p^2 dp \frac{p^2}{2M_n}}{\int_0^{p_F} p^2 dp} \\ &= \frac{\frac{1}{2M_n} \int_0^{p_F} p^4 dp}{\frac{1}{3} p^3 \Big|_0^{p_F}} \\ &= \frac{\frac{1}{2M_n} \frac{1}{5} p^5 \Big|_0^{p_F}}{\frac{1}{3} p_F^3} \\ &= \frac{\frac{1}{2M_n} \frac{1}{5} p_F^5}{\frac{1}{3} p_F^3} \\ &= \frac{1}{2M_n} \frac{3}{5} p_F^2 \end{aligned}$$

Part d

Substitute p_F from part (b) into part (c) to show that the average kinetic energy of a neutron is

$$\langle E_{kin} \rangle = \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9\pi N}{4} \right)^{2/3}$$

From part (b) we have $N = \frac{32}{9} \frac{\pi^2 R^3 p_F^3}{h^3}$. Solving for p_F , we get:

$$\begin{aligned} N &= \frac{32}{9} \frac{\pi^2 R^3 p_F^3}{h^3} \\ \frac{9h^3 N}{32} &= \pi^2 R^3 p_F^3 \\ \frac{9h^3 N}{32\pi^2 R^3} &= p_F^3 \\ \left(\frac{9h^3 N}{32\pi^2 R^3} \right)^{1/3} &= p_F \end{aligned}$$

Substituting this back into $\langle E_{kin} \rangle = \frac{1}{2M_n} \frac{3}{5} p_F^2$, we get:

$$\begin{aligned} \langle E_{kin} \rangle &= \frac{1}{2M_n} \frac{3}{5} p_F^2 \\ &= \frac{1}{2M_n} \frac{3}{5} \left(\left(\frac{9h^3 N}{32\pi^2 R^3} \right)^{1/3} \right)^2 \\ &= \frac{1}{2M_n} \frac{3}{5} \left(\frac{9h^3 N}{32\pi^2 R^3} \right)^{2/3} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2M_n} \frac{3}{5} \left(\frac{81h^6 N^2}{32^2 \pi^4 R^6} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{h^2}{M_n R^2} \left(\frac{81N^2}{32^2 \pi^4} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{(2\pi\hbar)^2}{M_n R^2} \left(\frac{81N^2}{32^2 \pi^4} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} 4\pi^2 \left(\frac{81N^2}{32^2 \pi^4} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{64 * 81N^2 \pi^6}{32^2 \pi^4} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{81N^2 \pi^2}{16} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9^2 N^2 \pi^2}{4^2} \right)^{\frac{1}{3}} \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}}
\end{aligned}$$

Part e

The average potential energy from gravitational attraction is

$$\langle E_{pot} \rangle = -\frac{3}{5} \frac{GNM_n^2}{R}$$

Minimize the total energy $\langle E_{tot} \rangle = \langle E_{kin} \rangle + \langle E_{pot} \rangle$ as a function of R to show that the equilibrium radius of the star is

$$R = \frac{\hbar^2 (9\pi/4)^{2/3}}{GM_n^3 N^{1/3}}$$

Evaluate this expression numerically (the answer is close to the measured value of 12 km).

If the total energy is $\langle E_{tot} \rangle = \langle E_{kin} \rangle + \langle E_{pot} \rangle$, then we have:

$$\begin{aligned}
\langle E_{tot} \rangle &= \langle E_{kin} \rangle + \langle E_{pot} \rangle \\
&= \frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} - \frac{3}{5} \frac{GNM_n^2}{R}
\end{aligned}$$

The minimum (or maximum) energy at a radius R is going to occur at a point when the derivative relative to R is equal to zero. So, differentiating we get:

$$\begin{aligned}
\frac{d}{dR} \langle E_{tot} \rangle &= \frac{d}{dR} \left(\frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} \right) - \frac{d}{dR} \left(\frac{3}{5} \frac{GNM_n^2}{R} \right) \\
&= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} \frac{d}{dR} \left(\frac{1}{R^2} \right) - \frac{3}{5} GNM_n^2 \frac{d}{dR} \left(\frac{1}{R} \right) \\
&= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (-2) \frac{1}{R^3} + \frac{3}{5} GNM_n^2 \frac{1}{R^2}
\end{aligned}$$

Setting this equal to zero, we get:

$$\begin{aligned}
 0 &= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (-2) \frac{1}{R^3} + \frac{3}{5} GNM_n^2 \frac{1}{R^2} \\
 \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (2) \frac{1}{R^3} &= \frac{3}{5} GNM_n^2 \frac{1}{R^2} \\
 \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} &= GNM_n^2 R \\
 \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} &= R \\
 \frac{\hbar^2 \left(\frac{9\pi}{4} \right)^{\frac{2}{3}}}{GN^{1/3} M_n^3} &= R
 \end{aligned}$$

However, this is only a minimum if the function is concave up at R . So, taking the second derivative, we get:

$$\begin{aligned}
 \frac{d^2}{dR^2} \langle E_{tot} \rangle &= \frac{d^2}{dR^2} \left(\frac{3}{10} \frac{\hbar^2}{M_n R^2} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} \right) - \frac{d^2}{dR^2} \left(\frac{3}{5} \frac{GNM_n^2}{R} \right) \\
 &= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} \frac{d^2}{dR^2} \left(\frac{1}{R^2} \right) - \frac{3}{5} GNM_n^2 \frac{d^2}{dR^2} \left(\frac{1}{R} \right) \\
 &= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (-2) \frac{d}{dR} \left(\frac{1}{R^3} \right) + \frac{3}{5} GNM_n^2 \frac{d}{dR} \left(\frac{1}{R^2} \right) \\
 &= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (6) \frac{1}{R^4} - \frac{6}{5} GNM_n^2 \frac{1}{R^3}
 \end{aligned}$$

Substituting in our minimum, we get:

$$\begin{aligned}
 \frac{d^2}{dR^2} \langle E_{tot} \rangle &= \frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (6) \frac{1}{R^4} - \frac{6}{5} GNM_n^2 \frac{1}{R^3} \\
 &= \frac{\frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (6)}{\left(\frac{\hbar^2 \left(\frac{9\pi}{4} \right)^{\frac{2}{3}}}{GN^{1/3} M_n^3} \right)^4} - \frac{\frac{6}{5} GNM_n^2}{\left(\frac{\hbar^2 \left(\frac{9\pi}{4} \right)^{\frac{2}{3}}}{GN^{1/3} M_n^3} \right)^3} \\
 &= \frac{\frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} (6)}{\frac{\hbar^8 \left(\frac{9\pi}{4} \right)^{\frac{8}{3}}}{G^4 N^{4/3} M_n^{12}}} - \frac{\frac{6}{5} GNM_n^2}{\frac{\hbar^6 \left(\frac{9\pi}{4} \right)^2}{G^3 N M_n^9}} \\
 &= \frac{\frac{3}{10} \frac{\hbar^2}{M_n} \left(\frac{9N\pi}{4} \right)^{\frac{2}{3}} G^8 N^{4/3} M_n^{12} (6)}{\hbar^8 \left(\frac{9\pi}{4} \right)^{\frac{8}{3}}} - \frac{\frac{6}{5} GNM_n^2 G^3 N M_n^9}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} \\
 &= \frac{\frac{3}{10} N^{2/3} G^4 N^{4/3} M_n^{11} (6)}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} - \frac{\frac{6}{5} GNM_n^2 G^3 N M_n^9}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} \\
 &= \frac{1}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} \left(\frac{3}{10} N^2 G^4 M_n^{11} (6) - \frac{6}{5} GNM_n^2 G^3 N M_n^9 \right) \\
 &= \frac{1}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} \left(\frac{9}{5} N^2 G^4 M_n^{11} - \frac{6}{5} G^4 N^2 M_n^{11} \right) \\
 &= \frac{1}{\hbar^6 \left(\frac{9\pi}{4} \right)^2} \left(\frac{3}{5} N^2 G^4 M_n^{11} \right)
 \end{aligned}$$

Since we know that \hbar , π , N , G , and M_n are all positive values, the value of the second derivative of the function at R is positive, so the function is concave up, hence R , as calculated above, is a minimum. The value of R , using N calculated in part (a), is 10.78 km.

Problem 8.11

From the absorption spectrum it is determined that about one out of 10^6 hydrogen atoms in a certain star is in the first excited state, 10.2 eV above the ground state (other excited states can be neglected). What is the temperature of the star? (Take the ratio of statistical weights to be 4, as in Example 8-2, [Tipler & Llewellyn, p. 318].)

If one out of 10^6 hydrogen atoms in a star is in the first excited state, then the ratio of atoms in the first excited state (n_2) to atoms in the ground state (n_1), is $\frac{n_2}{n_1} = \frac{1}{10^6} = 10^{-6}$. If the ratio of the degeneracies of the two states is 4, that is $\frac{g_2}{g_1} = 4$, then we can calculate the temperature by substitution into:

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{Ag_2 e^{-E_2/kT}}{Ag_1 e^{-E_1/kT}} \\ 10^{-6} &= 4 \frac{e^{-E_2/kT}}{e^{-E_1/kT}} \\ 10^{-6} &= 4e^{-E_2/kT + E_1/kT} \\ 10^{-6} &= 4e^{(-E_2 + E_1)/kT} \\ \frac{10^{-6}}{4} &= e^{(-E_2 + E_1)/kT} \\ \ln\left(\frac{10^{-6}}{4}\right) &= \frac{(-E_2 + E_1)}{kT} \\ T &= \frac{(-E_2 + E_1)}{k \ln\left(\frac{10^{-6}}{4}\right)} \\ T &= \frac{(-10.2 \text{ eV} + 0)}{k \ln\left(\frac{10^{-6}}{4}\right)} \\ T &= \frac{(-10.2 \text{ eV})}{k \ln\left(\frac{10^{-6}}{4}\right)} \\ T &= 7786.3 \text{ K} \end{aligned}$$

Problem 8.12

The first excited rotational energy state of the H_2 molecule ($g_2 = 3$) is about 4×10^{-3} eV above the lowest energy state ($g_1 = 1$). What is the ratio of the numbers of molecules in these two states at room temperature (300 K)?

Similar to problem 8.11, we can use $\frac{n_2}{n_1} = \frac{Ag_2 e^{-E_2/kT}}{Ag_1 e^{-E_1/kT}}$ to calculate this ratio. This gives us:

$$\begin{aligned} \frac{n_2}{n_1} &= \frac{Ag_2 e^{-E_2/kT}}{Ag_1 e^{-E_1/kT}} \\ &= 3 \frac{e^{-E_2/kT}}{e^{-E_1/kT}} \\ &= 3e^{(-E_2 + E_1)/kT} \\ &= 3e^{(-4 \times 10^{-3} \text{ eV} + 0)/k(300 \text{ K})} \\ &= 2.57 \end{aligned}$$

Problem 8.23

A container at 300 K contains H_2 gas at a pressure of one atmosphere. At this temperature H_2 obeys the Boltzmann distribution. To what temperature must H_2 gas be cooled before quantum effects become important and the use of the Boltzmann distribution is no longer appropriate? (Hint: Equate the de Broglie wavelength at the average energy to the average spacing between molecules, using the ideal gas law to compute the density.)

Following the outline in [Tipler & Llewellyn, p332], we start with the condition that when $\lambda \ll \langle d \rangle$, then we can ignore quantum effects. But, we have that:

$$\begin{aligned}\lambda &= \frac{h}{p} \\ &= \frac{h}{\sqrt{2mE_k}} \\ &= \frac{h}{\sqrt{2m(3kT/2)}} \\ &= \frac{h}{\sqrt{3kmT}}\end{aligned}$$

And we know that the average separation between particles is $\langle d \rangle = (V/N)^{1/3}$. So, if $\frac{h}{\sqrt{3kmT}} \ll (V/N)^{1/3}$, or when $\frac{N}{V} \frac{h^3}{(3kmT)^{3/2}} \ll 1$, we can no longer ignore quantum effects. The problem, however, seems to imply that there is some critical temperature when $\lambda = \langle d \rangle$. So, using this, we have that:

$$\begin{aligned}\frac{N}{V} \frac{h^3}{(3kmT)^{3/2}} &= 1 \\ \frac{N}{V} \frac{h^3}{(3km)^{3/2}} &= T^{3/2} \\ \left(\frac{N}{V}\right)^{2/3} \frac{h^2}{3km} &= T\end{aligned}$$

To solve this, we need to know the initial N/V , and since both N and V are constant. But, this is just $PV = NkT$, or $\frac{P}{kT} = \frac{N}{V}$. Substituting this in, we get:

$$\left(\frac{P}{kT_i}\right)^{2/3} \frac{h^2}{3km} = T_f$$

Where T_i is the initial temperature, 300 K, and T_f is the final temperature. So, we get that:

$$\left(\frac{1 \text{ atm}}{k300 \text{ K}}\right)^{2/3} \frac{h^2}{3km} = T_f$$

Or, that quantum effects become important around $T_f \approx 0.267 \text{ K}$.

Problem 8.26

Like ^4He , the most common form of neon, ^{20}Ne , is a rare gas and the ^{20}Ne atoms have zero spin and hence are bosons. But, unlike helium, neon does not become superfluid at low temperatures. Show that this is to be expected by computing neon's critical temperature and comparing it with the element's freezing point of 24.5 K.

The critical temperature for neon can be computed by:

$$T_c = \frac{h^2}{2mk} \left[\frac{N}{2\pi (2.315) V} \right]^{2/3}$$

Noting that $\frac{N}{V}$ is just the density of liquid neon, or 1.207 g/mL. This gives us that $T_c = 0.869$ K. This is significantly lower than the freezing point of neon.