University of Kentucky, Physics 361
Solution to Problem Set #8

1. a) The H_2 molecule consists of two hydrogen atoms bound by their shared electrons. The two electron wavefunctions span the entire molecule and have the same spatial wavefunction; therefore they must have opposite spin by the Pauli exclusion principle. However, the protons can be treated like point particles, whose wavefunctions do not overlap, and thus can have any combination of spins.

b) The protons has spin $s_1 = \frac{1}{2}$ and $s_2 = \frac{1}{2}$. Taking all combinations of $m_{s_1} = \pm\frac{1}{2}$ and $m_{s_2} = \pm\frac{1}{2}$: $(m_{s_1}, m_{s_2}) = (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})$.

c) The number of substates listed above is 4.

d) To calculate the quantum numbers for $\vec{S} = \vec{S}_1 + \vec{S}_2$, the z-component of total spin is $S_z = S_{z1} + S_{z2}$ or $m_s\hbar = m_{s_1}\hbar + m_{s_2}\hbar$, so $m_s = 1, 0, 0, -1$, respectively, for the above states. The total spin is also quantized: $S^2 = \hbar^2 s(s+1)$, and for each value of s, $m_s = -s, ..., s$ in steps of 1. Thus three substates $m_s = 1, 0, -1$ belong to $s = 1$, the triplet, and the remaining state $m_s = 0$ must have $s = 0$, the singlet. Note: the $(s{=}0, m{=}0)$ and $(s{=}1, m{=}0)$ states are actually two independent superpositions of the above $(m_{s_1}{=}\frac{1}{2}, m_{s_2}{=}-\frac{1}{2})$ and $(m_{s_1}{=}-\frac{1}{2}, m_{s_2}{=}\frac{1}{2})$ states. In every case, $s_1 = s_2 = \frac{1}{2}$.

Looking at it another way, the total spin quantum number can range from $s = |s_1 - s_2|$ to $s = s_1 + s_2$ in steps of one. Thus there are two s-states: $s = \frac{1}{2} - \frac{1}{2} = 0$ and $s = \frac{1}{2} - \frac{1}{2} = 1$, with the m-substates listed above.

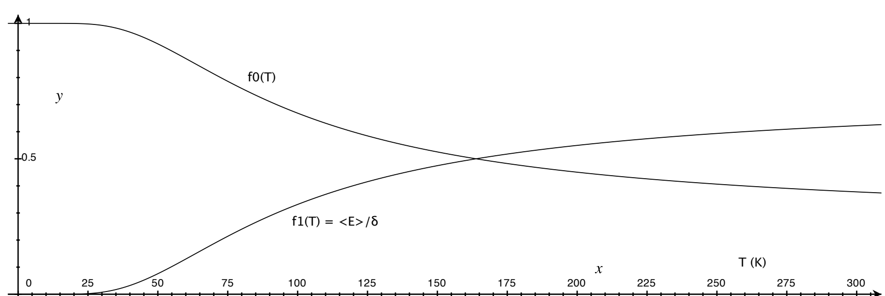
e) The $m = 1, -1$ states have $m_{s_1} = m_{s_2}$, or spins aligned, thus the tripet $s = 1$ states are orthohydrogen. Even $(s{=}1, m{=}0)$ state behaves as if the spins were aligned. The singlet state $s = 0$ is the lower energy parahydrogen. The degeneracies of these states are $g_1 = 3$ and $g_0 = 1$.

f) The number of molecules in each state $n_i = g_i f(\epsilon_i) = A g_i e^{-\epsilon_i/kT}$ where A is the normalization constant. Thus $n_0 = Ae^{-0/kT} = A$ and $n_1 = 3Ae^{-\delta/kT}$, where $\delta = 15$ meV and the total number of molecules is $n = A(1 + 3e^{-\delta/kT})$. The fractions or ortho- and para-hydrogen are:

$$f_0 \equiv \frac{n_0}{n} = \frac{1}{1 + 3e^{-\delta/kT}} \qquad \text{and} \qquad f_1 \equiv \frac{n_1}{n} = \frac{3e^{-\delta/kT}}{1 + 3e^{-\delta/kT}}.$$

g) The average energy is

$$\langle \epsilon \rangle = \frac{\sum_i \epsilon_i g_i f_i}{\sum_i g_i f_i} = \frac{0 n_0 + \delta n_1}{n_0 + n_1} = \frac{3\delta e^{-\delta/kT}}{1 + 3e^{-\delta/kT}} = f_1 \delta.$$



#2. a) $dN = dN_x \cdot dN_y \cdot dN_z = d^3p \cdot d^3x / h^3 = 4\pi p^2 dp \cdot V/h^3$

(integrating over angles of $\vec{p}$, and over space).

b) $\varepsilon = p^2/2m \qquad d\varepsilon = 2p\,dp/2m$

$dN = 4\pi V/h^3 \cdot p \cdot p\,dp = 4\pi V/h^3 \cdot \sqrt{2m\varepsilon} \cdot m\,d\varepsilon = g(\varepsilon)\,d\varepsilon.$

$g(\varepsilon) = 4\pi V/h^3 \cdot \sqrt{2m^3}\ \varepsilon^{1/2} d\varepsilon.$

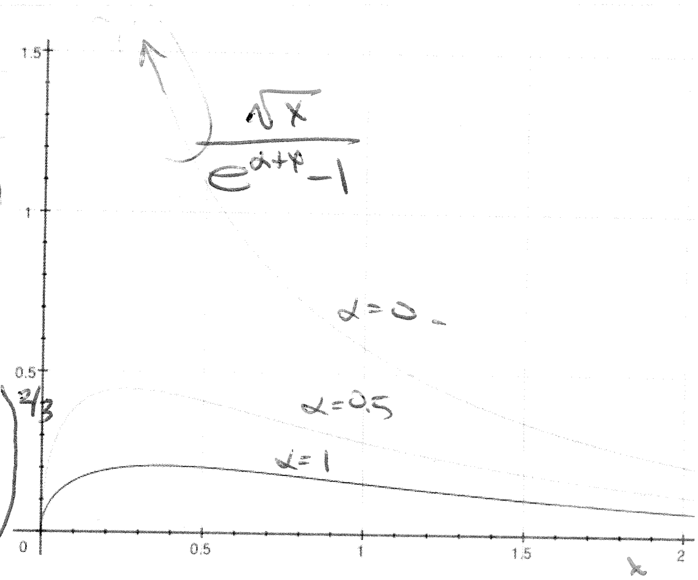
c) $N = \int_0^\infty g(\varepsilon) d\varepsilon \cdot f(\varepsilon) = \int_0^\infty 4\pi V/h^3 \cdot \sqrt{2m^3}\ \dfrac{\varepsilon^{1/2} d\varepsilon}{e^\alpha e^{\varepsilon/kT} - 1}$

let $x = \varepsilon/kT \qquad dx = \dfrac{d\varepsilon}{kT}$

$N = \int_{x=0}^\infty 4\pi V/h^3 \cdot \sqrt{2m^3 (kT)^3}\ \dfrac{\sqrt{x}\,dx}{e^{\alpha+x} - 1}$

$N/V = \dfrac{2\pi}{h^3} (2mkT)^{3/2}\, I(\alpha) \qquad$ where $I(\alpha) = \int_0^\infty \dfrac{\sqrt{x}\,dx}{e^{\alpha+x} - 1}$

d) $\longrightarrow$



e) $\dfrac{N}{V} = \dfrac{\rho N_A}{A} = \dfrac{2\pi}{h^3}(2mkT)^{3/2} I(\alpha)$

$T = \left(\dfrac{\rho N_A h^3}{2\pi A \cdot I(\alpha)}\right)^{2/3} \dfrac{1}{2mk}$

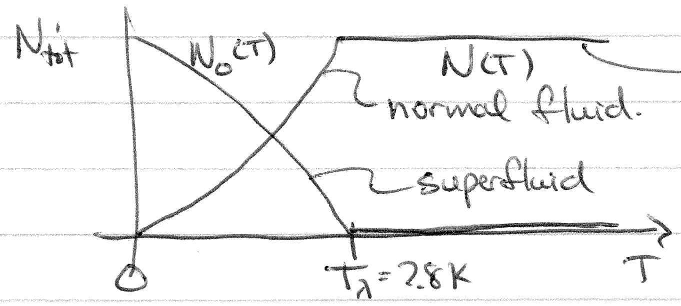
$= \dfrac{\left(\dfrac{0.125\,g/mL \cdot 6.02\times10^{23}/mol \cdot (1240\,eV\,nm/c)^3}{2\pi\ 4\,g/mol \cdot \{2.315, 0.7183, 0.3797\}}\right)^{2/3}}{2 \cdot 4 \cdot (0.938\,(GeV/c^2)) \cdot \dfrac{25\,meV}{300\,K}}$

$= 2.82\,K\ (\alpha=0),\ 6.15\,K\ (\alpha=0.5),\ 9.41\,K\ (\alpha=1)$

$\boxed{T_\lambda = 2.8\,K}$

f) $\dfrac{N}{N_\lambda} = \dfrac{\frac{2\pi}{h^3} V (2mkT)^{3/2}\, I(0)}{2\pi/h^3\, V(2mkT_\lambda)^{3/2}\, I(\alpha)} = \left(\dfrac{T}{T_\lambda}\right)^{3/2}$ if $T<T_\lambda$
(excited states)

$\dfrac{N_0}{N_\lambda} = 1 - \dfrac{N}{N_\lambda} = 1 - \left(\dfrac{T}{T_\lambda}\right)^{3/2}$ ground state



$I(\alpha)$ compensates for increasing temperature, keeping the # of particles fixed.

3. a) $N = \dfrac{1.5 m_\odot}{m_n} = \dfrac{1.5\times 1.9891\times10^{30}\,kg}{1.675\times10^{-27}\,kg} = 1.787\times10^{57}$ neutrons

b) $dN = 2\cdot(4\pi p^2 dp/h^3)V \quad \checkmark \quad = g(p)dp.$

c) $N = \int_0^\infty g(p)dp\, \underbrace{f(p)}_{\substack{1 \text{ if } p<p_F \\ 0 \text{ if } p>p_F}} = \int_0^{p_F} g(p)dp = \dfrac{8\pi}{h^3}V \int_0^{p_F} p^2 dp = \dfrac{8\pi p_F^3 V}{3h^3}$

d) $\langle E\rangle = \dfrac{\int g(p)dp\,\varepsilon(p)}{\int_0^{p_F} g(p)dp} = \dfrac{\int_0^{p_F} p^2dp\cdot\frac{p^2}{2M}}{\int_0^{p_F} p^2 dp} = \dfrac{\frac{1}{2M\cdot5}p_F^5}{\frac{1}{3}p_F^3} = \dfrac{3p_F^2}{10M_n}$

e) $\langle \varepsilon_{kin}\rangle = \dfrac{3p_F^2}{10m_n} = \dfrac{3}{10m_n}\left(\dfrac{3h^3\cdot N}{8\pi V}\right)^{2/3} =$ $\quad V=\frac{4}{3}\pi R^3$, $h = \hbar/2\pi$

$= \dfrac{3\hbar^2}{10M_nR^2}\left(\dfrac{9\pi N}{4}\right)^{2/3}$

f) $\dfrac{d}{dR}\left(\dfrac{3\hbar^2}{10M_nR^2}\cdot\left(\dfrac{9\pi N}{4}\right)^{2/3} - \dfrac{3}{5}\dfrac{GNM_n^2}{R}\right) = \dfrac{-3\hbar^2}{5M_nR^3}\left(\dfrac{9\pi N}{4}\right)^{2/3} + \dfrac{3}{5}\dfrac{GNM_n^2}{R^2} = 0$

$R = \dfrac{\hbar^2(9\pi/4)^{2/3}}{GM_n^3N^{1/3}} = \dfrac{(1.05\times10^{-34}\,J.s)^2\cdot\left(\frac{9\pi}{4}\right)^{2/3}}{6.67\times10^{-11}\,m^3/kg\,s^2\cdot(1.67\times10^{-27}\,kg)^3(1.8\times10^{57})^{1/3}} = \boxed{10.8\,km}$

#11: $\frac{n_2}{n_1} = \frac{g_2 e^{-E_2/kT}}{g_1 e^{-E_1/kT}} = 4e^{-\Delta/kT} = 4e^{-10.2eV/kT} = 10^{-6}$

$kT = 0.671 eV$ $\boxed{T = 7786 K}$

$k = \frac{25 meV}{300 K}$

ie: $kT = 25 meV$ @ room temperature

#12: $\frac{n_2}{n_1} = \frac{g_2 e^{-E_2/kT}}{g_1 e^{-E_1/kT}} = \frac{3}{1} e^{-\frac{4meV}{25meV}} = \boxed{2.56:1}$

#23: $\lambda << d$ $\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$ where $\langle \varepsilon \rangle = \langle \frac{p^2}{2m} \rangle = \frac{3}{2} kT$

$d = \left(\frac{V}{N}\right)^{1/3}$

so $\left(\frac{h^2}{3mkT}\right)^3 \approx \left(\frac{kT}{P}\right)^2$

$P_0 = 1 atm$

$T_0 = 300 K.$

if container is vacuum-tight so that V remains constant:

then $N/V = N/V_0 = \frac{kT_0}{P_0}$

$$kT = \frac{h^2}{3m}\left(\frac{P_0}{kT_0}\right)^{2/3} = k \cdot \boxed{\cancel{0.53}\ 2.67 K}$$

if the container is collapsible so that P remains constant:

then $\left(\frac{h^2}{3m}\right)^3 = \frac{(kT)^5}{P^2}$

$$kT = \left(\frac{h^2}{3m}\right)^{3/5} \cdot P_0^{2/5} = k \cdot \boxed{4.4 K}$$

alternate: $Nh^3 = V \cdot \frac{4}{3}\pi p_F^3$ $\langle \varepsilon \rangle = \frac{3p_F^2}{10m} = \frac{3}{2} kT$

$\frac{P}{kT} = \frac{4}{3}\pi\left(\frac{h^2}{5mkT}\right)^{3/2}$ $p_F^2 = 5mkT$

the same within a factor of 2.

#26. $T_c = \frac{h^2}{2mk}\left(\frac{\rho N_A}{2\pi I_0 A}\right)^{2/3} = 0.868 K.$

$<< 24.5 K.$

it freezes first

$A = 20.1797 g/mol$

$\rho = 1.207 g/cm^3$

$m = A/N_A$