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Fourier Transform of Gaussian Wave Packet

Gaussian wave packet: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Fourier transform of the Gaussian wave packet:

$$f(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{2\alpha}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-\alpha(k-k_0)^2} e^{ikx}\,dk$$

• use $k' = k - k_0 \Rightarrow dk' = dk$

$$f(x) = \left(\frac{\alpha}{2\pi^3}\right)^{1/4} e^{ik_0x}\int_{-\infty}^{\infty} e^{-\alpha(k-k_0)^2} e^{i(k-k_0)x}\,dk$$

$$= \left(\frac{\alpha}{2\pi^3}\right)^{1/4} e^{ik_0x}\int_{-\infty}^{\infty} e^{-\alpha k'^2} e^{ik'x}\,dk'$$

• use $k'' = k' - \frac{ix}{2\alpha}$ to get the term $e^{-\alpha(k'')^2}$, also $dk'' = dk' = dk$

Then $f(x) = \left(\frac{\alpha}{2\pi^3}\right)^{1/4} e^{ik_0x}\int_{-\infty}^{\infty} e^{-\alpha\left(k'-\frac{ix}{2\alpha}\right)^2} e^{-x^2/4\alpha}\,dk'$

$$= \left(\frac{\alpha}{2\pi^3}\right)^{1/4} e^{ik_0x} e^{-x^2/4\alpha}\int_{-\infty}^{\infty} e^{-\alpha(k'')^2}\,dk''$$

• note that $\int_{-\infty}^{\infty} e^{-\alpha(k'')^2}\,dk'' = \sqrt{\pi/\alpha}$

$$f(x) = \left(\frac{\alpha}{2\pi^3}\right)^{1/4}\sqrt{\frac{\pi}{\alpha}}\, e^{ik_0x} e^{-x^2/4\alpha}$$

$$\boxed{f(x) = \left(\frac{1}{2\pi\alpha}\right)^{1/4} e^{ik_0x} e^{-x^2/(4\alpha)}}$$

The RMS deviation (σx) is read from the Gaussian distn $P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$[f(x)]^2 = P(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\frac{-x^2}{2\alpha}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

so $\boxed{\sigma_x = \sqrt{\alpha}}$

In k-space, $P(k) = \sqrt{\frac{2\alpha}{\pi}}\, e^{-2\alpha(k-k_0)^2} \Rightarrow \boxed{\sigma_k = \frac{1}{2\sqrt{\alpha}}}$

Then $\sigma x(\sigma_k) = \sqrt{\alpha}\left(\frac{1}{2\sqrt{\alpha}}\right) = \frac{1}{2}$

Translating this into momentum (p), we get the limit (special case) of the Heisenberg Uncertainty Principle

$\boxed{\sigma x(\sigma_p) = \frac{1}{2}\hbar}$ [limit of Heisenberg Uncertainty principle]

Physically, we know that $\sigma x(\sigma_p)$ can always be much greater, so

[Heisenberg Uncertainty principle: $\sigma x(\sigma_p) \geq \frac{1}{2}\hbar$]