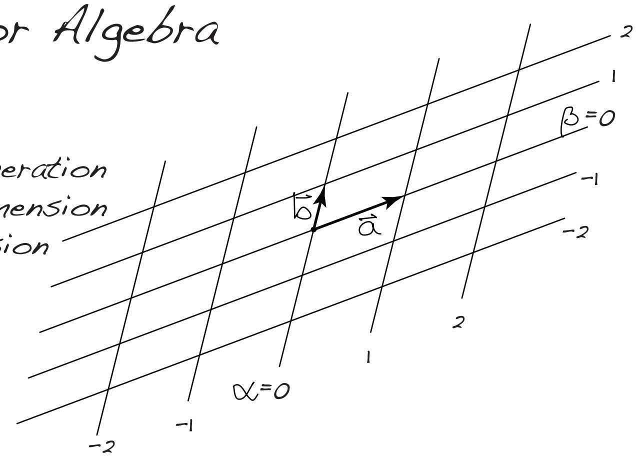
Section 1.1 – Vector Algebra



* Linear (vector) space

~ linear combination: $(\alpha\vec u+\beta\vec v)$ is the basic operation

~ basis: $(\hat x,\hat y,\hat z$ or $\vec a,\vec b,\vec c)$ # basis elements = dimension
  independence: not collapsed into lower dimension
  closure: vectors span the entire space

~ components: $\vec X=\vec a\alpha+\vec b\beta+\vec c\gamma=(\vec a\,\vec b\,\vec c)\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$
  in matrix form:

$$\vec X=\mathbb{b}\,\mathbb{X}$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}a_x & b_x & c_x\\ a_y & b_y & c_y\\ a_z & b_z & c_z\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

where

$$\vec a=\hat x a_x+\hat y a_y+\hat z a_z=(\hat x\,\hat y\,\hat z)\begin{pmatrix}a_x\\ a_y\\ a_z\end{pmatrix},\text{ etc}$$

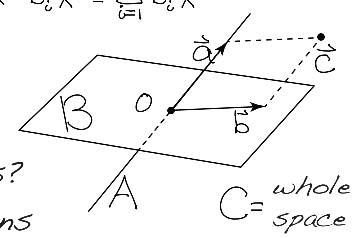
(usually one upper, one lower index)

~ Einstein notation: implicit summation over repeated indices $\quad \vec x=\vec b_i x^i\equiv\sum_{i=1}^{3}\vec b_i x^i$

~ direct sum: $C=A\oplus B$ add one vector from each independent space to get vector in the product space (not simply union)

~ projection: the vector $\vec c=\vec a+\vec b$ has a unique decomposition ('coordinates' $(\vec a,\vec b)$ in A,B) – relation to basis/components?

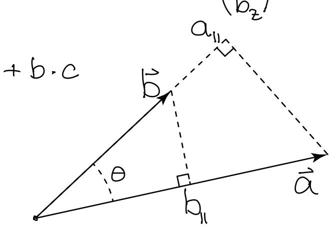
~ all other structure is added on as multilinear (tensor) extensions



* Metric (inner, dot product) – distance and angle

$$c=\vec a\cdot\vec b=ab\cos\theta=a_{\parallel}b=ab_{\parallel}=a_xb_x+a_yb_y+a_zb_z=a_ib^i=(a_x\,a_y\,a_z)\begin{pmatrix}b_x\\ b_y\\ b_z\end{pmatrix}$$

~ properties: 1) scalar valued – what is outer product?
  2) bilinear form $a\cdot(b+c)=a\cdot b+a\cdot c\quad (a+b)\cdot c=a\cdot c+b\cdot c$
  3) symmetric $\vec a\cdot\vec b=\vec b\cdot\vec a$



~ orthonormality and completeness – two fundamental identities help to calculate components, implicitly in above formulas

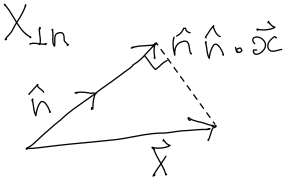
$$\boxed{\begin{array}{l}\hat e_i\cdot\hat e_j=\delta_{ij}\\ \sum_{i=1}^{3}\hat e_i\hat e_i\cdot=\mathbb{I}\end{array}}$$

Kroneker delta: components of the identity matrix

$$\delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\mathbb{I}$$

$$a^i=\vec a\cdot\hat e^i=a^1\hat e_1\cdot\hat e^i+a^2\hat e_2\cdot\hat e^i+a^3\hat e_3\cdot\hat e^i$$

~ orthogonal projection: a vector $\hat n$ divides the space X into $X_{\parallel n}\oplus X_{\perp n}$
  geometric view: dot product $\hat n\cdot\vec x$ is length of $\vec x$ along $\hat n$
  Projection operator: $P_{\parallel}\equiv\hat n\hat n\cdot$ acts on x: $P_{\parallel}\vec x=\vec x_{\parallel}=\hat n\,\hat n\cdot\vec x$



~ generalized metric: for basis vectors which are not orthonormal, collect all nxn dot products into a symmetric matrix (metric tensor)

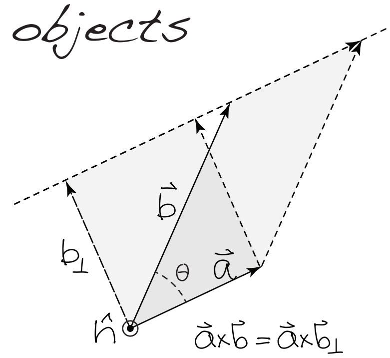
$$g_{ij}=\vec b_i\cdot\vec b_j\qquad \vec x\cdot\vec y=x^i\vec b_i\cdot\vec b_jy^j=x^ig_{ij}y^j$$

$$=\mathbb{X}^T\mathbb{b}^T\cdot\mathbb{b}\mathbb{y}=\mathbb{X}^T\mathbb{g}\,\mathbb{y}=(x^1\,x^2\,x^3)\overbrace{\begin{pmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{pmatrix}}^{g}\begin{pmatrix}y^1\\y^2\\y^3\end{pmatrix}$$

in the case of a non-orthonormal basis, it is more difficult to find components of a vector, but it can be accomplished using the reciprocal basis (see HW1)

Exterior Products - higher-dimensional objects

* cross product (area)



$$\vec{c} = \vec{a}\times\vec{b} = \hat{n}\, a b \sin\theta = \hat{n}\, a_\perp b = \hat{n}\, a b_\perp = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

where $\hat{n}\perp\vec{a}$ and $\hat{n}\perp\vec{b}$ (RH-rule)

$\vec{a}\times\vec{b} = \vec{a}\times\vec{b}_\perp$

$\vec{a}\times(\vec{b}-\vec{b}_\perp) = \vec{0}$ (parallel)

~ properties:
1) vector-valued
2) bilinear $a\times(b+c) = a\times b + a\times c \quad (a+b)\times c = a\times c + b\times c$
3) antisymmetric $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (oriented)

~ components: $\hat{e}_i \times \hat{e}_j = \varepsilon_{ij}{}^{k}\, \hat{e}_k$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$
$\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

Levi-Civita tensor - completely antisymmetric:

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ even permutation} \\ -1 & ijk \text{ odd permutation} \\ 0 & \text{repeated index} \end{cases}$$ (ijk cyclic)

$$\vec{x}\times\vec{y} = x^i \hat{e}_i \times \hat{e}_j y^j = \varepsilon_{ij}{}^{k}\, x^i y^j\, \hat{e}_k$$

~ orthogonal projection: $\hat{n}\times$ projects $\perp$ to $\hat{n}$ and rotates by 90°

$$\hat{x}_\perp = -\hat{n}\times(\hat{n}\times\hat{x}) = P_\perp \vec{x} \qquad P_\perp = -\hat{n}\times\hat{n}\times \qquad \boxed{P_\parallel + P_\perp = \hat{n}\hat{n}\cdot - \hat{n}\times\hat{n}\times = I}$$

~ where is the metric in $\times$?
vector $\times$ vector = pseudovector
symmetries act more like a 'bivector'
can be defined without metric

* triple product (volume of parallelpiped) - base times height

$$d = \vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

~ completely antisymmetric - definition of determinant
~ why is the scalar product symmetric / vector product antisymmetric?
~ vector vector $\times$ vector = pseudoscalar (transformation properties)
~ acts more like a 'trivector' (volume element)
~ again, where is the metric? (not needed!)

* exterior algebra (Grassman, Hamilton, Clifford)
~ extended vector space with basis elements from objects of each dimension
~ pseudo-vectors, scalar separated from normal vectors, scalar

magnitude,	length,	area,	volume
scalar,	vectors,	bivectors,	trivector
1,	$\hat{x}, \hat{y}, \hat{z}$,	$\hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}$,	$\hat{x}\hat{y}\hat{z}$

~ what about higher-dimensional spaces (like space-time)?
can't form a vector 'cross-product' like in 3-d, but still have exterior product

~ all other products can be broken down into these 8 elements
most important example: BAC-CAB rule (HW1: relation to projectors)

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$\varepsilon^i{}_{jk} A^j (\varepsilon^k{}_{mn} B^m C^n) = (\delta^i_m \delta_{jn} - \delta^i_n \delta_{jm}) A^j B^m C^n = B^i (A^j C_j) - C^i (A^j B_j)$$