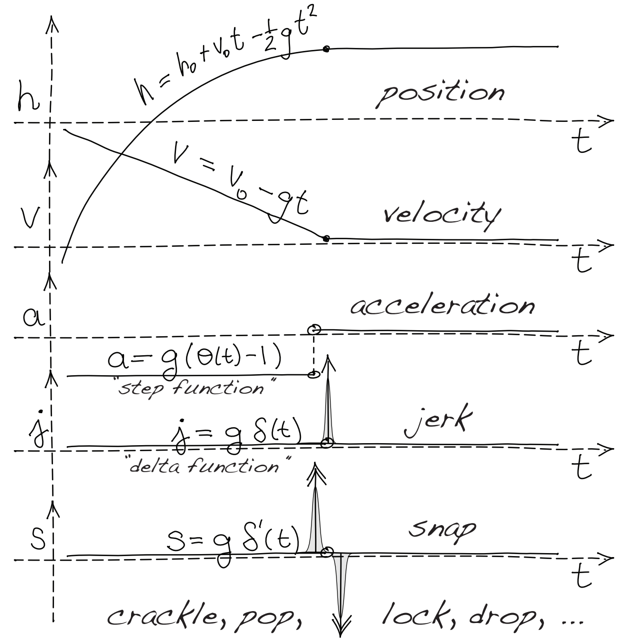
Section 1.5 – Dirac Delta Distribution

* Newton's law: yank = mass × jerk
http://wikipedia.org/wiki/position_(vector)



* definition: $d\Theta = \delta(x-x')dx$ is defined by its integral (a distribution, differential, or functional)

$$\int_a^b \underbrace{\delta(x)\,dx}_{d\Theta \text{ "differential"}} = \int_a^b d\Theta = \Theta(x)\Big|_a^b = \begin{cases} 1 & a<0<b \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$ it is a "distribution," NOT a function!

* important integrals related to $\delta(x)$

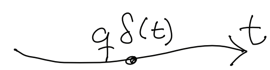
$$\int_{-\infty}^{\infty} \Theta(x)\, f(x)\, dx = \int_0^{\infty} f(x)\, dx \qquad \text{"mask"}$$

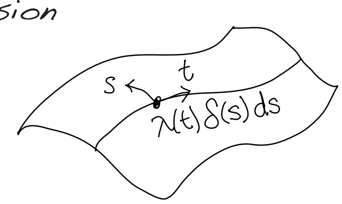
$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0) \qquad \text{"slit"}$$

$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx = -f'(0)$$

* $\delta(x-x')$ is the an "undistribution" – it integrates to a lower dimension

$$\int_C dq = \int_C \lambda\, dl = \int_C q \underbrace{\delta(t)\,dt}_{d\Theta} = q$$

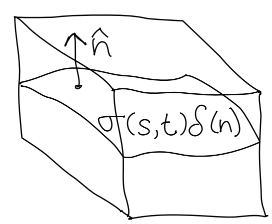




$$\int_A dq = \int_A \sigma\, da = \int_A \lambda(t) \underbrace{\delta(s)\,ds}_{d\Theta}\, dt = \int_C \lambda(t)\, dt = q$$

$$\int_V dq = \int_V \rho\, d\tau = \int_V \sigma(s,t) \underbrace{\delta(n)\,dn}\, ds\, dt = \int_A \sigma\, da = q$$

$$\text{or} = \int_V q\, \delta^3(\vec{r}) = q \qquad \text{or} = \int_V \lambda\, \delta^2(\vec{r}) = q$$



* $\delta(x-x')$ gives rise to boundary conditions – integrate the diff. eq. across the boundary

$$\nabla \cdot \vec{D} = \rho = \sigma(s,t)\,\delta(n)$$

$$\nabla \to \hat{n} \cdot \Delta \qquad \rho \to \sigma \qquad \vec{J} \to \vec{K}$$

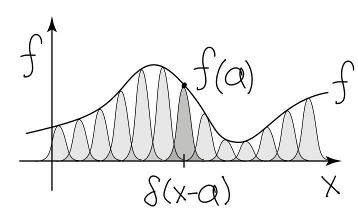
$$\int_{n=0^-}^{0^+} dn \left(\frac{\partial D_n}{\partial n} + \cancel{\frac{\partial D_s}{\partial s}} + \cancel{\frac{\partial D_t}{\partial t}} \right) = \int_{0^-}^{0^+} \sigma(s,t)\, \delta(n)\, dn$$

$$\boxed{\hat{n} \cdot \Delta \vec{D} = \sigma}$$

* $\delta(x-x')$ is the "kernel" of the identity transformation

$$f = I f \qquad f(x) = \underbrace{\int_{-\infty}^{\infty} dx'\, \delta(x-x')}_{\text{identity operator}} f(x')$$

(component form)



* $\delta(x-x')$ is the continuous version of the "Kroneker delta" δ_{ij}

$$a = I a \qquad a_i = \sum_{j=1}^{n} \delta_{ij}\, a_j \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

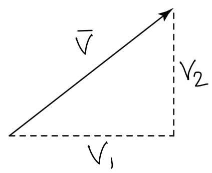
Linear Function Spaces

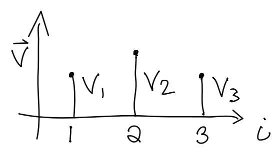
* *functions as vectors (Hilbert space)*

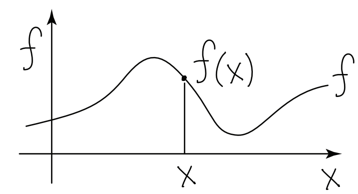
~ *functions under pointwise addition have the same linearity property as vectors*

	VECTORS	FUNCTIONS
~ *addition*	$\vec{w} = \vec{v} + \vec{u} \qquad w_i = v_i + u_i$	$h = f + g \qquad h(x) = f(x) + g(x)$
~ *expansion*	$\vec{v} = \sum_{\underbrace{i}_{\text{index}}} \underbrace{v_i}_{\text{component}} \underbrace{\hat{e}_i}_{\text{basis vector}} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + \ldots$	$f(x) = \int_{\underbrace{x'=-\infty}_{\text{index}}}^{\infty} \underbrace{f(x')}_{\text{component}} \cdot \underbrace{\delta(x-x')}_{\text{basis function}}$ or $f(x) = \sum_{i=0}^{\infty} \overbrace{f_i}^{} \cdot \overbrace{\phi_i(x)}^{}$

~ *graph*







	VECTORS	FUNCTIONS
~ *inner product (metric, symmetric bilinear product)*	$\vec{v} \cdot \vec{u} = \sum_{i=1}^{n} v_i u_i$	$\langle f \vert g \rangle = \int_{-\infty}^{\infty} dx\, f(x)\, g(x)$
~ *orthonormality (independence)*	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\int_{-\infty}^{\infty} \phi_i(x)\, \phi_j(x) = \delta_{ij} \qquad \int_{x'=-\infty}^{\infty} \delta(x-x')\delta(x'-y) = \delta(x-y)$
~ *closure (completeness)*	$\sum_{i=1}^{n} \hat{e}_i \hat{e}_i \cdot = I$	$\sum_{i=0}^{\infty} \phi_i(x)\, \phi_i(y) = \int_{x'=-\infty}^{\infty} \delta(x-x')\delta(x'-y) = \delta(x-y)$
~ *linear operator (matrix)*	$\vec{u} = A\vec{v} \qquad u_i = A_{ij} v_j$	$f = Hg \qquad f(x) = \int_{-\infty}^{\infty} dx'\, H(x,x')\, g(x')$
~ *orthogonal rotation (change of coordinates) (Fourier transform)*	$x' = Rx$ $R^T R = I$	$\tilde{f}(k) = \frac{1}{2\pi} \int dx\, e^{ikx} f(x)$ $\int dk\, e^{-ikx} e^{ikx'} = \int dk\, e^{-ik(x-x')} = 2\pi\, \delta(x-x')$
~ *eigen-expansion (stretches) (principle axes)*	$A\vec{v} = \vec{v}\lambda$ $AV = VW$	$H\phi(x) = \lambda \phi(x)$ *(Sturm-Liouville problems)*
~ *gradient, functional derivative*	$\nabla f = \frac{df}{d\vec{r}}$	$\frac{\delta F[\rho(x)]}{\delta \rho}$ *(functional minimization)*

* *Sturm-Liouville equation - eigenvalues of function operators (2nd derivative)*

$$\mathcal{L}[y] = -\frac{d}{dx}\left[p(x)\frac{d}{dx}y\right] + q(x) = \lambda\, w(x)\, y \qquad \text{BC: } y(a),\ y(b)$$

~ *there exists a series of eigenfunctions* $y_n(x)$ *with eigenvalues* λ_n

~ *eigenfunctions belonging to distinct eigenvalues are orthogonal* $\langle y_i | y_j \rangle = \delta_{ij}$