RESONANT FREQUENCY NEUTRON SPIN FLIPPER 5/10/11
DOUBLE COS-THETA COIL WINDING

INTRO

The purpose of this experiment is to accurately measure parity violation of protons. We will study the asymmetry associated with parity violation via a reaction of a neutron beam with a Helium-3 source, $n + {}^{3}He = {}^{3}H + {}^{1}H + 765\ keV$. In this reaction, a polarized, cold neutron beam is guided towards a Helium-3 target resulting in tritium, protium, and energy. The results of the reaction vary depending on the spin of the neutrons involved. We will collide beam after beam of either 'transverse up' or 'transverse down' neutrons with the Helium-3 target and compare the results, allowing the P.V. asymmetry to be precisely measured.

BASIC EXPERIMENTAL SETUP

The experiment starts with a highly accelerated proton beam colliding with a neutron spallation source. The resulting neutrons are then moderated in a liquid parahydrogen target so that they can be manipulated in the lab. At this point, we introduce a static, unidirectional magnetic field to experiment, which will polarize the neutrons either 'transverse up' or 'transverse down'. We will then guide the beam through a neutron supermirror which will absorb neutrons of 'down' spin, while allowing the passage of neutrons of 'up' spin. This ensures that we have a neutron beam which precesses in only one direction. The resultant beam passes through a chopper to reduce the range of velocities of the neutrons to be studied. The neutrons now pass through a resonant frequency spin flipper (RFSF). A power source will be connected to the apparatus in series along with a switchbox which will cycle between the RFSF and a resistive dummy load. This causes every other neutron beam to be flipped, allowing for a reaction of 'up' neutrons with the He-3 target to be closely compared with a reaction of 'down' neutrons with the He-3 target. The He-3 chamber contains several wires in a grid. We induce a large voltage drop across each wire with no current running through them. The walls of the chamber serve as a cathode, and the wires as anodes. The reaction of the neutrons with the He-3 target produces protons which, in turn, ionize the surrounding gas. Ions are attracted to the wire grid produces measurable current in the wires. This allows us to calculate momentum of the protons and measure the parity violation associated with the reaction.

POTENTIAL

First, we will solve for the two potentials for of our design. The first, inner potential will lie in the region $s = 0$ to $s = a$. The second potential will lie in the region $s = a$ to $s = A$. Additionally, we will require that the potential outside the two cylindrical shells is zero in all directions.

Calculating the potential inside the inner cylinder:

$$\vec{H} = -\nabla U = -\frac{\partial}{\partial x}(U) = H_o\hat{x}$$

Integrating:

$$-U = H_o x$$
$$U = -H_o x$$
$$U = -H_o s cos\phi$$

Calculating the potential between the inner and outer cylinder (solving Laplace's Eq.): $\nabla^2 \mathrm{U} = 0$

Note that the solution for potential does not depend on z.

General Solution: $U = a_o + b_o \ln(s) + \sum_{k=1}^{\infty} (a_k s^k + b_k s^{-k})(c_k cos(k\phi) + d_k \sin(k\phi))$

Boundaries:

@ $s = a$:	$U = -H_o s cos\phi$	(i)
@ $s = A$:	$U = 0$	(ii)

Due to boundary conditions, $a_o = d_k = 0$.

Also due to boundary conditions, we are only concerned with solutions in which $k = 1$.

Remaining Solution: $U = b_o \ln(s) + c_1 cos\phi(a_1 s + b_1 s^{-1})$

Let c_1 be absorbed by a_1 and b_1.

Remaining Solution: $U = b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})$

Differentiate each side with respect to s and evaluate at each boundary.

i)

$$\frac{\partial}{\partial s}(-H_o s cos\phi)|_{s=a} = \frac{\partial}{\partial s}[b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})]|_{s=a}$$
$$-H_o cos\phi = \left[\frac{b_o}{s} + cos\phi\left(a_1 - \frac{b_1}{s^2}\right)\right]|_{s=a}$$
$$-H_o cos\phi = \left[\frac{b_o}{a} + cos\phi\left(a_1 - \frac{b_1}{a^2}\right)\right]$$

Obviously, $b_o = 0$ and $-H_o = \left(a_1 - \frac{b_1}{a^2}\right)$

ii)

$$\frac{\partial}{\partial s}(0)|_{s=A} = \frac{\partial}{\partial s}[b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})]|_{s=A}$$
$$0 = \left[0 + cos\phi\left(a_1 - \frac{b_1}{s^2}\right)\right]|_{s=A}$$
$$0 = a_1 - \frac{b_1}{A^2}$$
$$a_1 = \frac{b_1}{A^2}$$

Plugging this back into i):

$$-H_o = \left(\frac{b_1}{A^2} - \frac{b_1}{a^2}\right)$$
$$-H_o = b_1\left(\frac{1}{A^2} - \frac{1}{a^2}\right)$$
$$-H_o = b_1\left(\frac{a^2}{A^2 a^2} - \frac{A^2}{A^2 a^2}\right)$$
$$b_1 = -\left(\frac{A^2 a^2}{a^2 - A^2}\right) H_o = \left(\frac{A^2 a^2}{A^2 - a^2}\right) H_o$$

Going back to our original equation for potential (k=1 solution):

$$U = a_o + b_o \ln(s) + (a_1 s + b_1 s^{-1})(c_1 cos\phi + d_1 \sin\phi)$$

With variables:

$a_o = b_o = d_k = 0$

c_1 absorbed.

$b_1 = \left(\frac{A^2a^2}{A^2-a^2}\right) H_o$

$a_1 = \left(\frac{a^2}{A^2-a^2}\right) H_o$

For the potential between the inner and outer cylinder, we have:

$$U = cos\phi\left(\left(\frac{a^2}{A^2-a^2}\right) H_o s + \left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1}\right)$$

$$U = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1})$$

SURFACE CURRENTS

Surface currents will run along equipotentials and help to visualize current flow throughout the two cylindrical shells.

Magnetostatic Boundary Condition: $\left[{}^{\partial}/_{\partial t}(U_2) - {}^{\partial}/_{\partial t}(U_1)\right]\hat{s} - \left[{}^{\partial}/_{\partial s}(U_2) - {}^{\partial}/_{\partial s}(U_1)\right]\hat{t} = \vec{K}$

Note: Coordinate system defined as $< \hat{n}, \hat{s}, \hat{t} >$

Note: U_2 defined as the potential in the region that the positive normal points from the surface. U_1 defined as the potential in the other region.

i) Inner Radial Surface

Normal: $\hat{s}$

Tangents: $\hat{\phi}$ and $\hat{z}$

$U_2 = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1})$

$U_1 = -H_o s cos\phi$

$$\left[{}^{\partial}/_{\partial z}(U_2) - {}^{\partial}/_{\partial z}(U_1)\right]\hat{\phi} - \left[{}^{\partial}/_{s\,\partial \phi}(U_2) - {}^{\partial}/_{s\,\partial \phi}(U_1)\right]\hat{z} = K_\phi\hat{\phi} + K_z\hat{z}$$

$K_\phi = \left[{}^{\partial}/_{\partial z}(U_2) - {}^{\partial}/_{\partial z}(U_1)\right]|_{s=a}$

$K_\phi = 0$ (since neither U_2 or U_1 depend on z)

$$K_z = -\left[{}^{\partial}/_{s\,\partial \phi}\left(\left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1})\right) - {}^{\partial}/_{s\,\partial \phi}(-H_o s cos\phi)\right]|_{s=a}$$

$$K_z = -\left[{}^{\partial}/_{\partial \phi}\left(\left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(1 + A^2 s^{-2})\right) - {}^{\partial}/_{\partial \phi}(-H_o cos\phi)\right]|_{s=a}$$

$$K_z = -\left[\left(-\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi(1 + A^2 a^{-2})\right) - (H_o sin\phi)\right]$$

$$K_z = H_o sin\phi\left[\left(\frac{a^2}{A^2-a^2}\right)(1 + A^2 a^{-2}) + 1\right]$$

$$K_z = H_o sin\phi\left[\left(\frac{a^2 + a^2 A^2 a^{-2}}{A^2-a^2}\right) + \frac{A^2-a^2}{A^2-a^2}\right]$$

$$K_z = H_o sin\phi\left(\frac{A^2+A^2}{A^2-a^2}\right)$$

$$K_z = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi$$

$$\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

ii) Outer Radial Surface
Normal: $\hat{s}$
Tangents: $\hat{\phi}$ and $\hat{z}$

$$U_2 = 0$$
$$U_1 = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1})$$

$$\left[\partial/_{\partial z}(U_2) - \partial/_{\partial z}(U_1)\right]\hat{\phi} - \left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]\hat{z} = K_\phi\hat{\phi} + K_z\hat{z}$$

$$K_\phi = \left[\partial/_{\partial z}(U_2) - \partial/_{\partial z}(U_1)\right]|_{s=A}$$
$K_\phi = 0$ (since neither U_2 or U_1 depend on z)

$$K_z = -\left[\partial/_{s\,\partial\phi}\left(\left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1})\right) - \partial/_{s\,\partial\phi}(0)\right]|_{s=A}$$
$$K_z = -\left[-\partial/_{\partial\phi}\left(\left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(1 + A^2 s^{-2})\right)\right]|_{s=A}$$
$$K_z = -\left[\left(\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi(1 + A^2 A^{-2})\right)\right]$$
$$K_z = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi$$

$$\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

iii) Inner End-Cap Surface (z = b)
Normal: $\hat{z}$
Tangents: $\hat{s}$ and $\hat{\phi}$

$$U_2 = 0$$
$$U_1 = -H_o s cos\phi$$

$$\left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]\hat{s} - \left[\partial/_{\partial s}(U_2) - \partial/_{\partial s}(U_1)\right]\hat{\phi} = K_s\hat{s} + K_\phi\hat{\phi}$$

$$K_s = \left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]$$
$$K_s = \left[\partial/_{s\,\partial\phi}(0) - \partial/_{s\,\partial\phi}(-H_o s cos\phi)\right]$$
$$K_s = -\partial/_{\partial\phi}(-H_o cos\phi)$$
$$K_s = -H_o sin\phi$$

$$K_\phi = -\left[\partial/_{\partial s}(U_2) - \partial/_{\partial s}(U_1)\right]$$
$$K_\phi = -\left[\partial/_{\partial s}(0) - \partial/_{\partial s}(-H_o s cos\phi)\right]$$
$$K_\phi = \partial/_{\partial s}(-H_o s cos\phi)$$
$$K_\phi = -H_o cos\phi$$

$$\vec{K} = -H_o sin\phi\ \hat{s} - H_o cos\phi\ \hat{\phi}$$

iv) Inner End-Cap Surface (z =- b)

Normal: $\hat{z}$
Tangents: $\hat{s}$ and $\hat{\phi}$

$$U_2 = -H_o s\cos\phi$$
$$U_1 = 0$$

$$\left[{}^{\partial}/_{s\,\partial\phi}(U_2) - {}^{\partial}/_{s\,\partial\phi}(U_1)\right]\hat{s} - \left[{}^{\partial}/_{\partial s}\,(U_2) - {}^{\partial}/_{\partial s}(U_1)\right]\hat{\phi} = K_s\hat{s} + K_\phi\hat{\phi}$$

Since U_2 and U_1 have simply switched from iii), we can say:

$$\vec{K} = H_o\sin\phi\,\hat{s} + H_o\cos\phi\,\hat{\phi}$$

v) Outer End-Cap Surface (z = b)

Normal: $\hat{z}$
Tangents: $\hat{s}$ and $\hat{\phi}$

$$U_2 = 0$$
$$U_1 = \left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(s + A^2 s^{-1})$$

$$\left[{}^{\partial}/_{s\,\partial\phi}(U_2) - {}^{\partial}/_{s\,\partial\phi}(U_1)\right]\hat{s} - \left[{}^{\partial}/_{\partial s}\,(U_2) - {}^{\partial}/_{\partial s}(U_1)\right]\hat{\phi} = K_s\hat{s} + K_\phi\hat{\phi}$$

$$K_s = \left[{}^{\partial}/_{s\,\partial\phi}(U_2) - {}^{\partial}/_{s\,\partial\phi}(U_1)\right]$$
$$K_s = \left[{}^{\partial}/_{s\,\partial\phi}(0) - {}^{\partial}/_{s\,\partial\phi}\left(\left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(s + A^2 s^{-1})\right)\right]$$
$$K_s = -{}^{\partial}/_{\partial\phi}\left(\left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(1 + A^2 s^{-2})\right)$$
$$K_s = \left(\frac{a^2}{A^2-a^2}\right)H_o\sin\phi(1 + A^2 s^{-2})$$

$$K_\phi = -\left[{}^{\partial}/_{\partial s}\,(U_2) - {}^{\partial}/_{\partial s}(U_1)\right]$$
$$K_\phi = -\left[{}^{\partial}/_{\partial s}\,(0) - {}^{\partial}/_{\partial s}\left(\left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(s + A^2 s^{-1})\right)\right]$$
$$K_\phi = {}^{\partial}/_{\partial s}\left(\left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(s + A^2 s^{-1})\right)$$
$$K_\phi = \left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(1 - A^2 s^{-2})$$

$$\vec{K} = \left(\frac{a^2}{A^2-a^2}\right)H_o\sin\phi(1 + A^2 s^{-2})\,\hat{s} + \left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(1 - A^2 s^{-2})\,\hat{\phi}$$

vi) Outer End-Cap Surface (z =- b)

Since U_2 and U_1 have simply switched from v), we can say:

$$\vec{K} = -\left(\frac{a^2}{A^2-a^2}\right)H_o\sin\phi(1 + A^2 s^{-2})\,\hat{s} - \left(\frac{a^2}{A^2-a^2}\right)H_o\cos\phi(1 - A^2 s^{-2})\,\hat{\phi}$$

<u>BASIC DESIGN</u>

The design of the RFSF is based upon a cos-theta winding pattern. It is composed of 2 natural nylon shells, one inside the outer, surrounded by an aluminum cylindrical shell which acts as magnetic shielding. Natural nylon is chosen for the inner two shells because it is non-conductive and, thus, will not affect the magnetic field created by the coil.

ALUMINUM MAGNETIC SHIELD

The skin depth of aluminum is calculated to ensure that is will be thick enough to act as a magnetic shield for the RFSF.

In normal cases, $\delta = \sqrt{\frac{2\rho}{\omega\mu}}$,

For Aluminum: $\rho = resistivity = 2.82\times10^{-8}\Omega \cdot m$
$\omega = frequency = 183247.185\ Hz$
$\mu = permittivity = 1.2566650\times10^{-6}\ \mathrm{H}/\mathrm{m}$

Thus, $\delta = 0.00049489\ m = 0.019484\ in$

For the experiment, an aluminum cylindrical shell of $0.375\ in \approx 19.25 \cdot \delta$ thickness is chosen to ensure that no magnetic field leaks from the RFSF into any other portion of the experiment.

Due to material availability, the inner diameter of the aluminum shell is $15.5\ in$, which correlates to an outer diameter of $16.25\ in$ (which provides for the $0.375\ in$ wall). It's length is $15.75\ in$.

NYLON SHELLS DIMENSIONS

The non-conductive nylon shells will serve as placeholders for the wire windings of the cos-theta coil. We want to leave $0.005\ in$ of 'play' between the aluminum shell and the outer nylon shell. Therefore, the outer diameter of the outer nylon shell is $15.4900\ in$. The wire windings here are completely inset into the nylon material. The standard diameter of 18 gauge wire is $1.024\ mm$. However, in actuality, copper wire has a very thin coating of protective material on it. We estimate the realistic diameter of 18 gauge wire to be $\approx 1.1mm \approx 0.043307\ in$. The center of the wires on this surface will form the $s = A$ boundary.

Thus, $2A = \left(15.49 - 2\left(\frac{0.043307}{2}\right)\right) in$
$A = 7.723346\ in$

We require that the wires at the $s = a$ boundary to be spaced to meet the condition $\Delta x_{in} = 4\Delta x_{out}$.

$U_{in} = n_{in}\Delta \mathrm{I} = \mathrm{H_o x_n} = \mathrm{H_o} n\ \Delta \mathrm{x_{in}}$
$U_{out} = n\Delta \mathrm{I} = \left(\frac{a^2}{A^2-a^2}\right) \mathrm{H_o x_n}(1 + \mathrm{A^2 s^{-2}}) = \left(\frac{a^2}{A^2-a^2}\right) \mathrm{H_o} n\ \Delta \mathrm{x_{out}}(1 + \mathrm{A^2 s^{-2}})$

Since ΔI doesn't change throughout the coil, we can compare the two potentials to solve for the optimal value of inner boundary, a, for the RFSF.

$\mathrm{H_o n\ \Delta x_{in}} = \left(\frac{a^2}{A^2-a^2}\right) \mathrm{H_o n\ \Delta x_{out}}(1 + \mathrm{A^2 s^{-2}})$
$4 = \left(\frac{a^2}{A^2-a^2}\right)\left(1 + \frac{\mathrm{A}^2}{\mathrm{a}^2}\right)$
$4 = \left(\frac{A^2+a^2}{A^2-a^2}\right)$
$4(A^2-a^2) = A^2 + a^2$
$3A^2 = 5a^2$
$a = \sqrt{\frac{3}{5}}\ A = 5.982478\ in$

On this boundary, the wires will be halfway inset into both the inner nylon shell and the outer nylon shell. We also want to leave $0.006\ in$ of 'play' between the two nylons cylinders, divided evenly between them. Therefore, the inner diameter of the outer nylon shell will be $2a + 2(0.003\ in) = 11.97096\ in$ and the outer diameter of the inner nylon shell will be $2a - 2(0.003\ in) = 11.95896\ in$. The only condition requirement of the inner nylon shell is that it is durable. Thus, the wall of the inner nylon shell will be 0.5 in thick, correlating to an inner diameter of $(11.95896 - 1)in = 10.95896\ in$.

The length of the nylon cylinders in the RFSF are chosen to be $\approx 0.125\ in$ shorter than the aluminum shell such that wire windings around the nylon end-caps will be confined within the length of the aluminum shell. Thus, their lengths will be $\approx 15.625\ in$, which also provides an approximate 1:1 ratio of length to outer diameter for the RFSF.

<u>REVIEW OF SPECS</u>

Aluminum:	O.D.:	16.25 in
	I.D.:	15.5 in
	Length:	15.75 in
Outer Nylon:	O.D.:	15.49 in
	I.D.:	11.97096 in
	Length:	15.625 in
Inner Nylon:	O.D.:	11.95896 in
	I.D.:	10.95896 in
	Length:	15.625 in

<u>FLIPPING THE NEUTRON</u>

The RFSF will sit in a static magnetic field with constant strength of $\vec{B} = B_o\hat{x} = 10G\ \hat{x}$. Now, we must match the Larmor frequency of a neutron such that it completes a full rotation from the "up" to "down" position.

$Larmor\ frequency = \omega_L = -\gamma B$ where $\gamma = {}^{g_N \mu_N}/_{\hbar}$

For a neutron: $g_N = -3.82608545$ (correction in quantum mechanics)
$\mu_N = 5.05078324x10^{-27}\ {}^{J}/_{T}$ (nuclear magneton for a neutron)
$\hbar = reduced\ plank's\ constant = 1.054571628\ x\ 10^{-34}\ J \cdot s$

$\therefore\ \omega_L = 183.247185\ kHz$

$f_L = {}^{W_L}/_{2\pi} = 29.1646953\ kHz$

To calculate the velocity of the neutron, we focus on neutrons with a de Broglie wavelength of $\lambda = 5\text{Å}$.

$p = m_n v = {}^{h}/_{\lambda}$

For a neutron: $m_n = mass\ of\ a\ neutron = 1..67492729\ x\ 10^{-27}\ kg$
$h = plank's\ constant = 6.2606896\ x\ 10^{-34}\ J \cdot s$

$\therefore v = \frac{h}{m_n \lambda} = 791.206644\ {}^{m}/_{s}$

Given this velocity, the neutron will pass through the RFSF in $\Delta t = \frac{L}{v}$, where L is the distance between the center of the wires on the nylon end-caps. Thus, $L \approx \left(\left(\frac{15.625+15.75}{2}\right)\right)\ in = 15.6875\ in$. Therefore, $\Delta t = 503.614\ \mu s$.

Since the neutron will flip from 0 radians (up) to π radians (down), the rate of flip of the neutron needs to be $\omega_F = \frac{\pi}{\Delta t} = 6.2381001\ kHz$.

Next, we need to calculate the rotational magnetic field required to induce this rate of flip in the neutron so that it will complete only a single, complete flip during its time in the RFSF.

$$\omega_F = -\gamma B_{rot} = 6.2381001\ kHz$$

$$\therefore\ B_{rot} = -\frac{\omega_F}{\gamma} = 0.3404199\ G$$

From the neutron's perspective, this is the maximum value of the magnetic field needed for it to complete one rotation. In reality, we need to generate twice this amount of magnetic field. Therefore, the resonant frequency field needed for the neutron spin flipper is $B_{RF} = 2B_{rot}e^{i\omega t}$ and will have a maximum value of $B_o = 2B_{rot} = 0.6808399\ G$.

In order to compare this to potential, we convert the B-field to its H-field counterpart:

$$H_o = {}^{B_o}/_{\mu_o} = 54.179521\ {}^{A}/_{m}$$

WIRE WINDINGS AND INDUCTANCE

To maximize the number of RFSF windings, and thus create the most uniform magnetic field possible, there will be 52 wire windings around the inner nylon cylindrical shell. According to our 4:1 winding ratio requirement, there will be 416 windings around the outer cylinder (208 per outer half). Ideally, this will establish a spacing of $\Delta x_{in} = 4\Delta x_{out}$ at the $s = a$ boundary. In actuality, the outer wire placements on this boundary are adjusted to create an equal spacing between all wire points (inner and outer). In both the inner and outer cylinders, wires will run along evenly spaced equipotentials for maximum result.

Inner Cylinder Wire Placement

We first look at the inner wire points on the $s = a$ boundary. Since the wires will wrap around the inner cylinder such that they are parallel with the y-axis, we have $\Delta x_{in} = {}^{0.3039099\ m}/_{52} = 0.00584448\ m$. After the adjustment mentioned above, the smallest value of Δx anywhere on the RFSF will be $\frac{\Delta x_{in}}{5} = 0.0011689\ m$, which is enough space to allow for the winding of 18 gauge wire.

The maximum current depends on Δx_{in}.

$$U = H_o x_n = n\Delta I \text{ where } x_n = n\ \Delta x$$
$$\therefore\ \Delta I = H_o \Delta x = 316.647945\ mA$$

Also, the x, y wire point coordinates on the end-caps of the RFSF are known due to Δx_{in}(See Appendix A).

Outer Cylinder Wire Placement

Now we look at the outer wire points on the $s = a$ boundary. Remember that we require four times the number of outer wire points compared to inner points on this boundary. Note that we also want an equal Δx between all wire points (inner and outer) at the $s = a$ boundary. Thus, the $s = a$ wire placements are slightly shifted on the outer nylon shell to accomplish this (See Appendix B, Part I).

We next need to determine the value of Δx_{out} on the boundary $s = A$ (See Appendix B, Part II).

Here, $U = n\Delta I = \left(\frac{a^2}{A^2-a^2}\right) H_o x_n \left(1 + {}^{A^2}/_{s^2}\right)$ where $x_n = n\Delta x$

$$\therefore\ \Delta I = 2\left(\frac{a^2}{A^2-a^2}\right) H_o \Delta x_{out}$$

$$\therefore\ \Delta x_{out} = \frac{\Delta I}{2\left(\frac{a^2}{A^2-a^2}\right)H_o} = 0.00194814\ m$$

Now that the wire spacing on the boundary $s = A$ is known, calculations for the x and y coordinates can be made on this boundary.

It's important to note here that not all end-cap wire segments that originate on the $s = a$ boundary converge to the $s = A$ boundary. This is due to the path of equipotentials which originate close to the x-axis in the outer nylon shell. These few equipotentials cross the x-axis and converge back onto the $s = a$ boundary. To emulate this, we will place pegs on the x-axis of the outer shell to run wires along. To calculate their positions:

i) $U(s = a) = U_a = known$

ii) $U(x, y = 0) = U_A = \left(\frac{a^2}{A^2-a^2}\right) H_o x \left(1 + {A^2}/{x^2}\right)$

$$U_a = U_A = \left(\frac{a^2}{A^2-a^2}\right) H_o x \left(1 + {A^2}/{x^2}\right)$$

$$x\left(1 + {A^2}/{x^2}\right) = \frac{U_a}{\left(\frac{a^2}{A^2-a^2}\right)H_o}$$

$$x - \frac{U_a}{\left(\frac{a^2}{A^2-a^2}\right)H_o} + {A^2}/{x} = 0$$

$$x^2 - \left(\frac{U_a}{\left(\frac{a^2}{A^2-a^2}\right)H_o}\right)x + A^2 = 0$$

Solving this via the quadratic formula: $x = \dfrac{\frac{U_a}{\left(\frac{a^2}{A^2-a^2}\right)H_o} \pm \sqrt{\left(\frac{U_a}{\left(\frac{a^2}{A^2-a^2}\right)H_o}\right)^2 + 4A^2}}{2}$

This formula gives the value of x on the x-axis for equipotentials (and thus wire segments), which start at the $s = a$ boundary and do not converge to the $s = A$ boundary.

INDUCTANCE

To calculate the inductance of the inner windings, we must first calculate the flux through each wire loop such that $\Phi = (\Phi_1 + \Phi_2 + \ldots + \Phi_i) = \Delta I(\mathrm{l}_1 + l_2 + \ldots + \mathrm{l}_i)$, where $(\mathrm{l}_1 + l_2 + \ldots + \mathrm{l}_i) = L$, the total inductance inside.

The flux through any single loop of the inner cylinder is $\Phi_\mathrm{i} = \iint_\mathrm{S} \vec{\mathrm{B}} \cdot \overrightarrow{\mathrm{dS}} = \iint_\mathrm{S} \mu_o \vec{\mathrm{H}} \cdot \overrightarrow{\mathrm{dS}}$. The surface of each of these loops runs in the yz-plane and the normal to the surface points in the same direction as the H-field. Therefore, $\iint_\mathrm{S} \mu_o \vec{\mathrm{H}} \cdot \overrightarrow{\mathrm{dS}}$ becomes $\iint_\mathrm{S} \mu_o \mathrm{H}\ \mathrm{dS}$. Since the H-field is constant across the surface, we have $\Phi_\mathrm{i} = \mu_o \mathrm{H_o} \iint_\mathrm{S} \mathrm{dS} = \mu_o \mathrm{H_o}\ \mathrm{A_i}$, where $\mathrm{A_i}$ is simply the area of the loop. Using this, we can calculate flux running through each loop. The sum is $\Phi_\mathrm{in} = 0.00034232$ webers. The corresponding inductance for the inner coil is $L_{in} = 1.081064\ mH$.

To calculate the inductance of the loops around the outer cylinder, we first have to look at the flux through these loops. Notice that all of the flux through the top half of the inner cylinder runs entirely through the top half of the outer cylinder. The ratio of wires out to in is exactly $\frac{208}{52} = 4$. The same goes for the bottom half of the flipper as well. Therefore, $\Phi_\mathrm{out} = 4\Phi_\mathrm{in} = 0.00136927$ webers. The corresponding inductance for the outer coil is $L_{out} = 4L_{in} = 4.324256\ mH$.

The total inductance of the coil will be $L_{tot} = L_{in} + L_{out} = 5.405319\ mH$.

The total inductance can also be approximated by integrating over the nylon cylinders.

In General: $L = \frac{2E}{I^2}$ and $E = \frac{\mu_o}{2} \iiint_V \left|\vec{H}\right|^2 dV$

Since $\vec{H}$ doesn't depend on z, $L = \frac{\mu_o z}{I^2} \iint_S \left|\vec{H}\right|^2 dS$

i) Inner Nylon Cylinder

$$L = \frac{\mu_o}{I^2} z \iint_S H_o{}^2 dS$$
$$L = \frac{\mu_o}{I^2} z\, H_o{}^2 \int_0^a \int_0^{2\pi} s\, ds d\phi$$
$$L = \frac{\mu_o}{I^2} 2\pi z\, H_o{}^2 \int_0^a s\, ds$$
$$L = \frac{\mu_o}{I^2} \pi a^2 z\, H_o{}^2$$

ii) Outer Nylon Cylinder

$$L = \frac{\mu_o}{I^2} z \iint_S \left[\left(\frac{a^2}{A^2-a^2}\right) H_o \left[cos\phi\left(1-\frac{A^2}{s^2}\right)\hat{s} - sin\phi\left(1+\frac{A^2}{s^2}\right)\hat{\phi}\right]\right]^2 dS$$
$$L = \frac{\mu_o}{I^2} z \int_a^A \int_0^{2\pi} \left[\left(\frac{a^2}{A^2-a^2}\right)^2 H_o{}^2 \left[cos^2\phi\left(1-\frac{A^2}{s^2}\right)^2 + sin^2\phi\left(1+\frac{A^2}{s^2}\right)^2\right]\right] sdsd\phi$$
$$L = \frac{\mu_o}{I^2} \pi z \left(\frac{a^2}{A^2-a^2}\right)^2 H_o{}^2 \int_a^A \left[\left(1-\frac{A^2}{s^2}\right)^2 + \left(1+\frac{A^2}{s^2}\right)^2\right] sds$$
$$L = \frac{\mu_o}{I^2} \pi z \left(\frac{a^2}{A^2-a^2}\right)^2 H_o{}^2 \left[\left(A^2 - \frac{A^4}{A^2}\right) - \left(a^2 - \frac{A^4}{a^2}\right)\right]$$
$$L = \frac{\mu_o}{I^2} \pi z \left(\frac{a^2}{A^2-a^2}\right)^2 H_o{}^2 \left[\left(\frac{A^4}{a^2} - a^2\right)\right]$$

	Summation	Integration	Percent Difference
Inner	1.081064 mH	1.059157 mH	1.64803%
Outer	4.324256 mH	4.253573 mH	1.64803%
Total	5.405319 mH	5.316966 mH	1.64803%

CAPACITANCE

The capacitance required to drive the circuit can be calculated from the inductance of the coil.

$L = \frac{1}{\omega^2 C}$ where $\omega = \omega_L = 183.247185\ kHz$

$\therefore\ C = \frac{1}{\omega^2 L} = 5.509396\ nF$

We will choose a capacitor to match this value and connect it in series with the circuit. This will allow for minimal driving power of the RLC circuit over time.

WIRE GAUGE AND RESISTANCE

Our smallest value of Δx on the three boundaries above is $\Delta x = 0.0011689\ m$ (occurs on the $s = a$ boundary after slightly shifting the wire placements here). As mentioned before, 18 gauge wire fits in this space comfortably with a diameter of 0.001024 m (+thin coating).

The resistance of the coil can be determined from the wire gauge. 18 gauge copper wire has a resistance of 0.02095 Ω/m. We multiply this by the total length of wire inside to get a resistance of $R_{in} = 1.385198\ \Omega$

(Appendix A). Repeating the process for the outer loops, we get $R_{out} = 4.230995\ \Omega$ (Appendix B). This includes those wires which do not converge to $s = A$ from $s = a$ on the end-caps.

The total resistance of the coil is $R_{tot} = R_{in} + R_{out} = 5.616194\ \Omega$.

QUALITY FACTOR

We calculate the quality factor of the circuit as a measure of its efficiency. It compares the ratio of the coil's inductive reactance to its resistance at the resonant frequency.

$$Q = \frac{\omega L}{R} = 176.3667$$

POWER

We need to know the power required to fully charge the capacitor over $400\mu s$. We also need to know the driving power of the circuit after this occurs.

Reactance

Resistance: $X_R = R = 5.616194\ \Omega$

Inductive Reactance: $X_L = \omega L = 990.509587\ \Omega$

Capacitive Reactance: $X_C = -\frac{1}{\omega C} = -990.509587\ \Omega$

Energy stored in the RLC circuit will slosh back and forth between the capacitor and inductor after the circuit has been fully charged.

Energy Stored

Inductor: $E_L = \frac{1}{2} L I^2 = 270.984669\ \mu J$

Capacitor: $E_C = \frac{1}{2} C V^2 = 270.984669\ \mu J$

Slosh Energy

Slosh: $E_{slosh} = E_L = E_C = 270.984669\ \mu J$

The ramp power (power required to fully charge the capacitor in $400\mu s$) is the sum of the power lost to resistance and the power needed to charge the capacitor.

Real Power (power lost to resistance): $P_{real} = \frac{1}{2} I^2 R = 0.281556\ W$

Charge Power (power to charge capacitor): $P_{charge} = \frac{E_C}{\Delta t} = 0.677642\ W$

Ramp Power: $P_{ramp} = P_{real} + P_{charge} = 0.95901809\ W$

So, the RFSF will be held at P_{ramp} for the first $400\mu s$ and at P_{real} thereafter. The RLC circuit will then operate at driven, damped harmonic resonance.

REROUTE DESIGN POTENTIAL

This design removes the inner nylon cylindrical shell entirely, which is preferred. To do this and still maintain our desired magnetic field, we need to reroute the inner potential around through the outer loops.

Calculating the potential to be rerouted (solving Laplace's Eq.): $\nabla^2 \mathrm{U} = 0$

Note that the solution for potential does not depend on z.

General Solution: $U = a_o + b_o \ln(\mathrm{s}) + \sum_{k=1}^{\infty} (a_k s^k + b_k s^{-k})\big(c_k cos(k\phi) + d_k \sin(k\phi)\big)$

Boundaries: @ $s = a$: $U = -(-H_o s cos\phi)$ (i) (across $s = a$ boundary)

@ $s = A$: $U = 0$ (ii)

Due to boundary conditions, $a_o = d_k = 0$.
Also due to boundary conditions, we are only concerned with solutions in which $k = 1$.

Remaining Solution: $U = b_o \ln(s) + c_1 cos\phi(a_1 s + b_1 s^{-1})$

Let c_1 be absorbed by a_1 and b_1.

Remaining Solution: $U = b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})$

Compare each side at the boundaries:

i) $H_o s cos\phi|_{s=a} = [b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})]|_{s=a}$

$$H_o a cos\phi = \left[b_o \ln(a) + cos\phi\left(a_1 a + \frac{b_1}{a}\right)\right]$$

Obviously, $b_o = 0$

$$H_o a = a_1 a + \frac{b_1}{a}$$
$$a_1 a = H_o a - \frac{b_1}{a}$$
$$a_1 = H_o - \frac{b_1}{a^2}$$

ii) $0|_{s=A} = [b_o \ln(s) + cos\phi(a_1 s + b_1 s^{-1})]|_{s=A}$

$$0 = \left[0 + cos\phi\left(a_1 A + \frac{b_1}{A}\right)\right]$$
$$0 = a_1 A + \frac{b_1}{A}$$
$$a_1 = -\frac{b_1}{A^2}$$

Plugging this back into i):

$$-\frac{b_1}{A^2} = H_o - \frac{b_1}{a^2}$$
$$H_o = -\frac{b_1}{A^2} - \frac{b_1}{a^2}$$
$$b_1\left(-\frac{a^2}{A^2 a^2} - \frac{A^2}{A^2 a^2}\right) = H_o$$
$$b_1 = -\left(\frac{A^2 a^2}{a^2 - A^2}\right) H_o = \left(\frac{A^2 a^2}{A^2 - a^2}\right) H_o$$

Going back to our original equation for potential (k=1 solution):

$$U = a_o + b_o \ln(\mathrm{s}) + (a_1 s + b_1 s^{-1})(c_1 cos\phi + d_1 \sin\phi)$$

With variables:

$a_o = b_o = d_k = 0$

c_1 absorbed.

$$b_1 = \left(\frac{A^2 a^2}{A^2 - a^2}\right) H_o$$
$$a_1 = -\left(\frac{a^2}{A^2 - a^2}\right) H_o$$

For the rerouted potential through the outer cylinder, we have:

$$U_{reroute} = cos\phi\left(-\left(\frac{a^2}{A^2 - a^2}\right) H_o s + \left(\frac{A^2 a^2}{A^2 - a^2}\right) H_o s^{-1}\right)$$
$$U_{reroute} = \left(\frac{a^2}{A^2 - a^2}\right) H_o cos\phi(-s + A^2 s^{-1})$$

The total outer potential will now be:

$$U = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1}) + U_{reroute}$$
$$U = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1}) + \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(-s + A^2 s^{-1})$$
$$U = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(s + A^2 s^{-1} - s + A^2 s^{-1})$$
$$U = \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi(2A^2 s^{-1})$$
$$U = 2\left(\frac{A^2 a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi$$

REROUTE WINDINGS

Windings can be calculated as before for the outer nylon shell. Their positions will change with the new design to compensate for the removal of the inner loops. Wire points that start at the $s = a$ boundary will simply converge to different points on the $s = A$ boundary. This allows the same grooved nylon cylinders to be used for both designs, whether or not the inner nylon cylindrical shell is indcluded. Appendix C lists the winding points on one end-cap.

SURFACE CURRENTS: OUTER ONLY

Magnetostatic Boundary Condition: $\left[{}^{\partial}/_{\partial t}(U_2) - {}^{\partial}/_{\partial t}(U_1)\right]\hat{s} - \left[{}^{\partial}/_{\partial s}(U_2) - {}^{\partial}/_{\partial s}(U_1)\right]\hat{t} = \vec{K}$

Note: Coordinate system defined as $< \hat{n}, \hat{s}, \hat{t} >$

Note: U_2 defined as the potential in the region that the positive normal points from the surface. U_1 defined as the potential in the other region.

i) Inner Radial Surface

Normal: $\hat{s}$

Tangents: $\hat{\phi}$ and $\hat{z}$

$$U_2 = 2\left(\frac{A^2 a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi$$
$$U_1 = 0$$

$$\left[{}^{\partial}/_{\partial z}(U_2) - {}^{\partial}/_{\partial z}(U_1)\right]\hat{\phi} - \left[{}^{\partial}/_{s\,\partial\phi}(U_2) - {}^{\partial}/_{s\,\partial\phi}(U_1)\right]\hat{z} = K_\phi\hat{\phi} + K_z\hat{z}$$

$$K_\phi = \left[{}^{\partial}/_{\partial z}(U_2) - {}^{\partial}/_{\partial z}(U_1)\right]|_{s=a}$$
$K_\phi = 0$ (since neither U_2 or U_1 depend on z)

$$K_z = -\left[{}^{\partial}/_{s\,\partial\phi}\left(2\left(\frac{A^2 a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi\right) - 0\right]|_{s=a}$$
$$K_z = -\left[{}^{\partial}/_{\partial\phi}\left(2\left(\frac{A^2 a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\right)\right]|_{s=a}$$
$$K_z = 2\left(\frac{A^2 a^2}{A^2-a^2}\right) H_o a^{-2} sin\phi$$
$$K_z = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi$$

$$\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

ii) Outer Radial Surface

Normal: $\hat{s}$

Tangents: $\hat{\phi}$ and $\hat{z}$

$$U_2 = 0$$

$U_1 = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi$

$\left[\partial/_{\partial z}(U_2) - \partial/_{\partial z}(U_1)\right]\hat{\phi} - \left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]\hat{z} = K_\phi\hat{\phi} + K_z\hat{z}$

$K_\phi = \left[\partial/_{\partial z}(U_2) - \partial/_{\partial z}(U_1)\right]|_{s=a}$
$K_\phi = 0$ (since neither U_2 or U_1 depend on z)

$K_z = -\left[0 - \partial/_{s\,\partial\phi}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi\right)\right]|_{s=A}$
$K_z = \left[\partial/_{\partial\phi}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\right)\right]|_{s=A}$
$K_z = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o A^{-2} sin\phi$
$K_z = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi$

$\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$

iii) Outer End-Cap Surface (z = b)

Normal: $\hat{z}$
Tangents: $\hat{s}$ and $\hat{\phi}$

$U_2 = 0$
$U_1 = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi$

$\left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]\hat{s} - \left[\partial/_{\partial s}(U_2) - \partial/_{\partial s}(U_1)\right]\hat{\phi} = K_s\hat{s} + K_\phi\hat{\phi}$

$K_s = \left[\partial/_{s\,\partial\phi}(U_2) - \partial/_{s\,\partial\phi}(U_1)\right]$
$K_s = \left[\partial/_{s\,\partial\phi}(0) - \partial/_{s\,\partial\phi}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi\right)\right]$
$K_s = -\partial/_{\partial\phi}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\right)$
$K_s = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} sin\phi$

$K_\phi = -\left[\partial/_{\partial s}(U_2) - \partial/_{\partial s}(U_1)\right]$
$K_\phi = -\left[\partial/_{\partial s}(0) - \partial/_{\partial s}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi\right)\right]$
$K_\phi = \partial/_{\partial s}\left(2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-1} cos\phi\right)$
$K_\phi = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi$

$\vec{K} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} sin\phi\ \hat{s} - 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\ \hat{\phi}$

iv) Outer End-Cap Surface (z =- b)
Since U_2 and U_1 have simply switched from iii), we can say:

$\vec{K} = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} sin\phi\ \hat{s} + 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\ \hat{\phi}$

COMSOL CONVERSIONS

I. Design: Inner & Outer Shell

a. Outer Radial: $\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$

b. Inner Radial: $\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$

c. Inner End-Cap: $\vec{K} = -H_o sin\phi\ \hat{s} - H_o cos\phi\ \hat{\phi}$ $(z = b)$

d. Inner End-Cap: $\vec{K} = H_o sin\phi\ \hat{s} + H_o cos\phi\ \hat{\phi}$ $(z = -b)$

e. Outer End-Cap: $\vec{K} = \left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi(1 + A^2 s^{-2})\ \hat{s} + \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi \left(1 - \frac{A^2}{s^2}\right)\ \hat{\phi}$ $(z = b)$

f. Outer End-Cap: $\vec{K} = -\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi(1 + A^2 s^{-2})\ \hat{s} - \left(\frac{a^2}{A^2-a^2}\right) H_o cos\phi \left(1 - \frac{A^2}{s^2}\right)\ \hat{\phi}$ $(z = -b)$

II. Design: Outer Only

a. Outer Radial: $\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$

b. Inner Radial: $\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$

c. Outer End-Cap: $\vec{K} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} sin\phi\ \hat{s} - 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\ \hat{\phi}$ $(z = b)$

d. Outer End-Cap: $\vec{K} = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} sin\phi\ \hat{s} + 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o s^{-2} cos\phi\ \hat{\phi}$ $(z = -b)$

$\hat{\boldsymbol{s}} = \cos\varphi\hat{\boldsymbol{x}} + \sin\varphi\hat{\boldsymbol{y}}$

$\hat{\boldsymbol{\varphi}} = -\sin\varphi\hat{\boldsymbol{x}} + \cos\varphi\hat{\boldsymbol{y}}$

IN CARTESIAN:

I. Design: Inner & Outer

a. Inner Radial:

$$\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

$$\boldsymbol{\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o \frac{y}{s}\ \hat{z}}$$

b. Outer Radial:

$$\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

$$\boldsymbol{\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o \frac{y}{s}\ \hat{z}}$$

c. + Inner End-Cap:

$$\vec{K} = -H_o sin\phi\ \hat{s} - H_o cos\phi\ \hat{\phi} \quad (z = b)$$

$$\vec{K} = (-H_o sin\phi cos\phi + H_o sin\phi cos\phi)\hat{x} + (-H_o sin^2\phi - H_o cos^2\phi)\hat{y}$$

$$\boldsymbol{\vec{K} = -H_o\ \hat{y}}$$

d. – Inner End-Cap:

$$\vec{K} = H_o sin\phi\ \hat{s} + H_o cos\phi\ \hat{\phi}$$

$$\vec{K} = (H_o sin\phi cos\phi - H_o sin\phi cos\phi)\hat{x} + (H_o sin^2\phi + H_o cos^2\phi)\hat{y}$$

$$\boldsymbol{\vec{K} = +H_o\ \hat{y}}$$

e. + Outer End-Cap:

$$\vec{K} = \left(\frac{a^2}{A^2-a^2}\right) H_o \left[s\phi\left(1 + \frac{A^2}{s^2}\right)(c\phi\hat{x} + s\phi\hat{y}) + c\phi\left(1 - \frac{A^2}{s^2}\right)(-s\phi\hat{x} + c\phi\hat{y})\right]$$

$$K_x\hat{x} = \left(\frac{a^2}{A^2-a^2}\right) H_o \left[c\phi s\phi\left(1 + \frac{A^2}{s^2}\right) - c\phi s\phi\left(1 - \frac{A^2}{s^2}\right)\right]\hat{x}$$

$$K_x\hat{x} = \left(\frac{a^2}{A^2-a^2}\right) H_o \frac{xy}{s^2}\left[\left(1 + \frac{A^2}{s^2}\right) - \left(1 - \frac{A^2}{s^2}\right)\right]\hat{x}$$

$$\boldsymbol{K_x\hat{x} = \left(\frac{a^2}{A^2-a^2}\right) H_o \frac{xy}{s^2}\left[\left(\frac{2A^2}{s^2}\right)\right]\hat{x} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \frac{xy}{s^4}\hat{x}}$$

$$K_y\hat{y} = \left(\frac{a^2}{A^2-a^2}\right) H_o \left[s\phi s\phi\left(1 + \frac{A^2}{s^2}\right) + c\phi c\phi\left(1 - \frac{A^2}{s^2}\right)\right]\hat{y}$$

$$\boldsymbol{K_y\hat{y} = \left(\frac{a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)\left[y^2\left(1 + \frac{A^2}{s^2}\right) + x^2\left(1 - \frac{A^2}{s^2}\right)\right]\hat{y}}$$

f. - Outer End-Cap:

$$\vec{K} = -\left(\frac{a^2}{A^2-a^2}\right) H_o \left[s\phi\left(1+\frac{A^2}{s^2}\right)(c\phi\hat{x}+s\phi\hat{y}) + c\phi\left(1-\frac{A^2}{s^2}\right)(-s\phi\hat{x}+c\phi\hat{y})\right]$$

$$\boldsymbol{K_x\hat{x} = -\left(\frac{a^2}{A^2-a^2}\right) H_o \frac{xy}{s^2}\left[\left(\frac{2A^2}{s^2}\right)\right]\hat{x} = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \frac{xy}{s^4}\hat{x}}$$

$$\boldsymbol{K_y\hat{y} = -\left(\frac{a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)\left[y^2\left(1+\frac{A^2}{s^2}\right) + x^2\left(1-\frac{A^2}{s^2}\right)\right]\hat{y}}$$

II. Design: Outer Only

a. Inner Radial:

$$\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

$$\boldsymbol{\vec{K} = 2\left(\frac{A^2}{A^2-a^2}\right) H_o \frac{y}{s}\ \hat{z}}$$

b. Outer Radial:

$$\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o sin\phi\ \hat{z}$$

$$\boldsymbol{\vec{K} = -2\left(\frac{a^2}{A^2-a^2}\right) H_o \frac{y}{s}\ \hat{z}}$$

c. + Outer End-Cap:

$$\vec{K} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)[s\phi(c\phi\hat{x}+s\phi\hat{y}) - c\phi(-s\phi\hat{x}+c\phi\hat{y})]$$

$$K_x\hat{x} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)[2s\phi c\phi]\hat{x}$$

$$\boldsymbol{K_x\hat{x} = 4\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{xy}{s^4}\right)\hat{x}}$$

$$K_y\hat{y} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)[s\phi s\phi - c\phi c\phi]\hat{y}$$

$$\boldsymbol{K_y\hat{y} = 2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^4}\right)[y^2 - x^2]\hat{y}}$$

d. - Outer End-Cap:

$$\vec{K} = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^2}\right)[s\phi(c\phi\hat{x}+s\phi\hat{y}) - c\phi(-s\phi\hat{x}+c\phi\hat{y})]$$

$$\boldsymbol{K_x\hat{x} = -4\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{xy}{s^4}\right)\hat{x}}$$

$$\boldsymbol{K_y\hat{y} = -2\left(\frac{A^2a^2}{A^2-a^2}\right) H_o \left(\frac{1}{s^4}\right)[y^2 - x^2]\hat{y}}$$