PHY 611 – Electromagnetic Theory I
Final Exam – Saturday, December 8, 2012 – Due at 4:00 p.m.

Problem 1 [20 points]: Electrostatics
Consider a sphere of radius R with a spherically symmetric charge density which varies with r as $\rho(r) = Nr^{-1}$, where N is some constant. The total charge of the sphere is $Q > 0$.

(a) Calculate N. [2 points]
(b) Calculate the electric field $\vec{E}$ both inside and outside the sphere. [10 points]
(c) Make a rough sketch of $|\vec{E}|$ as a function of r. [3 points]
(d) Calculate the work that would be required (i.e., against the action of the $\vec{E}$ field) to move a point charge $Q' > 0$ from $r = \infty$ to $r = 0$. [5 points]

Problem 2 [30 points]: Magnetic Force and Energy
As shown below, a rectangular loop of wire carrying a constant current I_1 is placed near an infinitely long wire carrying a constant current I_2. The rectangular loop is centered at $(d, 0, 0)$ in the indicated coordinate system. Note that the direction of the current I_2 in the wire is the same as the direction of the current I_1 in the side of the rectangular loop which is closest to the wire. Assume both I_1 and I_2 have been constant since $t = -\infty$.

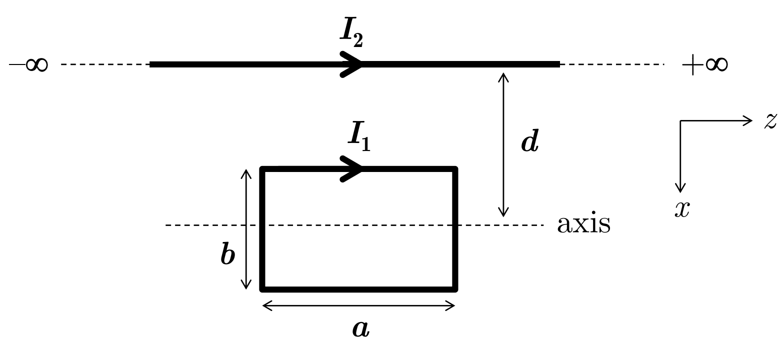
(a) Calculate the magnetic vector potential $\vec{A}_2(\vec{x})$ of the long wire starting from the definition of $\vec{A}$ as given in Eq. (5.32) of Jackson. *Hint*: In order to most efficiently next work part (b), you might consider writing any integrals of the form as

$$\int_{-\infty}^{+\infty} du\ f(u) = 2\int_{0}^{+\infty} du\ f(u)$$

if the symmetry of the integrand permits you to do so. Then, you might consider writing any integrals of the form as

$$\int_{0}^{+\infty} du\ f(u) = \lim_{L\to\infty}\int_{0}^{+L} du\ f(u)$$

and then proceed to evaluate the integral in terms of L. [10 points]
(b) Show that $\vec{B}_2(\vec{x}) = \vec{\nabla}\times\vec{A}_2(\vec{x})$ in the limit of $L \to \infty$ is as you would expect. [5 points]
(c) Calculate the two currents' magnetic interaction energy in the limit of $L \to \infty$. [15 points]



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Problem 3 [50 points]: Various Topics

Consider a solid conducting sphere of radius R with a total integral charge of zero. The sphere rotates with constant angular velocity ω about the z-axis, so that $\vec{\omega} = \omega\hat{z}$. We will assume $\omega R \ll c$ (i.e., so that the velocity everywhere in the sphere is $\ll c$, which implies we can employ Galilean transformations). Suppose a magnetic dipole moment $\vec{m} = m\hat{z}$ is located at the center of the sphere, and is fixed in position (i.e., it does not move).

(a) Write an expression for the $\vec{B}$ field of the magnetic dipole moment $\vec{m}$ in terms of spherical coordinates (r, θ, ϕ) and spherical unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ for $r > 0$. *You do not need to derive it from first principles; it is sufficient to identify the appropriate expression in Jackson and then express it in spherical coordinates and spherical unit vectors.* Give a physical argument for what the $\vec{B}'$ field in the sphere rest frame (i.e., in the reference frame co-rotating with the sphere) is in terms of the $\vec{B}$ field in the lab frame. [5 points]

(b) What is the electric field $\vec{E}'$ in the sphere rest frame (i.e., in the reference frame co-rotating with the sphere)? [3 points]

(c) Calculate the electric field $\vec{E}$ in the lab frame for $0 < r < R$. [7 points]

(d) What is the value of $\vec{\nabla} \times \vec{E}$ everywhere in space? [*Hint: Think before you attempt any type of calculation.*] Can $\vec{E}$ then, in principle, be written as $\vec{E} = -\vec{\nabla}\Phi$, where Φ is the electrostatic potential? *You do not need to find Φ here; a conceptual answer is sufficient.* [5 points]

(e) Calculate the voltage (i.e., the potential difference) between some point $(r = R, \theta > 0)$ on the surface of the sphere and its North Pole. [10 points]

$$\text{Voltage} = \Phi(r = R, \theta > 0) - \Phi(r = R, \theta = 0)$$

(f) Find the charge density $\rho(r, \theta)$ which is established inside of the sphere for $0 < r < R$. *Optional: You can check your answer by calculating the total integral charge.* [5 points]

(g) Find the potential $\Phi(r, \theta)$ outside of the sphere for $r > R$. [15 points]

Potentially Useful Formulas

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln\left(u + \sqrt{u^2 + a^2}\right)$$

$$\hat{x} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$$

$$\hat{y} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}$$

$$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$$

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta \qquad P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$

$$\sin^2\theta = \frac{2}{3}P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta)$$

End of Exam