PHY 611
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Lecture #1

- Course webpage, syllabus, Problem sets posted on webpage
  [www.pa.uky.edu/~plaster/phy611]
- will be challenging course; best way to learn the material will be to work the problem sets, but manageable; experimentalist, not theorist!
- Problem sets: 6–7, due at start of class on specified date
  - No late problem sets will be accepted

Goal: theory, & how to use it

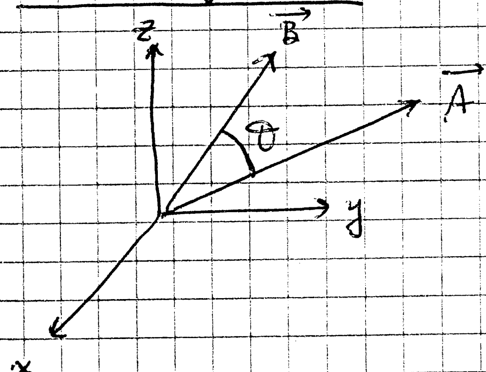
- Midterm Exam: evening exam
- Final Exam: evening/weekend exam
- Grading Policy
- Office Hours: open-door policy [PHY 151 right after this]

Note: have to leave promptly!

[Arfken Chapter 1]

Today: Mathematics Review

Properties of Vectors



Let's write these in rectangular coordinates as:

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

A_x, B_x : projections of $\vec{A}, \vec{B}$ on $\hat{x}$
⋮

(omit "Σ")
implied summation convention
sum over $\underbrace{\text{repeated indices}}$

$$= A_i B_i$$

Dot product defined to be:

$$\vec{A}\cdot\vec{B} \equiv A_x B_x + A_y B_y + A_z B_z = \sum_i A_i B_i = \sum_i B_i A_i \equiv \vec{B}\cdot\vec{A}$$

Note: if $\vec{A} = \vec{B}$: $\vec{A}\cdot\vec{A} = |\vec{A}|^2 = \sum_i A_i^2 \Rightarrow |\vec{A}| = \sqrt{\sum_i A_i^2}$

Now let θ = angle between $\vec{A}$ and $\vec{B}$:

Just like $A_x = \vec{A}\cdot\hat{x}$ (projection of $\vec{A}$ on $\hat{x}$)

We can think of $\vec{A}\cdot\vec{B}$ as:

(2)

$$\vec{A}\cdot\vec{B} = \vec{A}\cdot\left(|\vec{B}|\hat{B}\right) = |\vec{B}|\,\underbrace{\vec{A}\cdot\hat{B}}$$

projection of $\vec{A}$ on unit vector along $\vec{B}$, $\hat{B} = \dfrac{\vec{B}}{|\vec{B}|}$

By trigonometry: $\vec{A}\cdot\hat{B} = |\vec{A}|\cos\theta$

$$\Rightarrow \vec{A}\cdot\vec{B} = |\vec{B}|\,|\vec{A}|\cos\theta \qquad = A_i B_i$$

(index notation), sum over repeated indices (omit "$\sum$") on same side of equation "implied summation"

$$\Rightarrow \boxed{\vec{A}\cdot\vec{B} = \sum_i A_i B_i = |\vec{A}|\,|\vec{B}|\cos\theta} \quad \text{(scalar)}$$

$$\underbrace{(= B_i A_i)}_{\text{commutative}}$$

Cross Product: $\vec{C} = \vec{A}\times\vec{B}$ (vector)

$$C_i = \sum_{j,k}\underbrace{\varepsilon_{ijk}}_{\text{Levi-Civita Symbol}} A_j B_k \qquad \text{"index notation"}$$

Levi-Civita Symbol

anti-symmetric w.r.t. all pairs of indices

$$\begin{pmatrix} \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \\ \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \\ \text{all else } \varepsilon_{ijk} = 0 \end{pmatrix}$$

e.g.,

$$C_1 = \varepsilon_{123}A_2B_3 + \varepsilon_{132}A_3B_2 \qquad \text{all other } \varepsilon_{1jk} = 0$$

$(i=1)$

$$= A_2B_3 - A_3B_2 \quad ✓$$

Magnitude: $|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}|\sin\theta$

Anti-commutative: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Gradient Operator $\vec{\nabla}$:

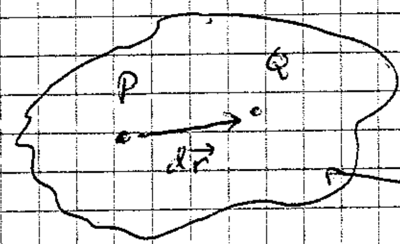
Suppose $\phi(x,y,z)$ is a scalar function [function whose value depends only on the coordinates (x,y,z)]

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}, \qquad \text{[vector]}$$

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Geometric Interpretation of the Gradient:

Consider two points P and Q on a surface defined by $\phi(x,y,z) = \text{constant}$



Q a distance $d\vec{r}$ from P

$\phi(x,y,z) = C$ ("equipotential")

In moving from P to Q:

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = 0 \quad \text{(by definition)}$$

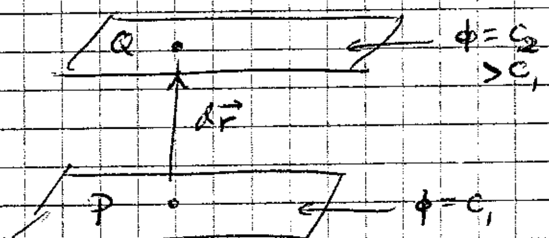
[Recall: $\vec{A}\cdot\vec{B} = 0$ if $\vec{A} \perp \vec{B}$] arbitrary P, Q ⇒

$\Rightarrow \vec{\nabla}\phi \perp d\vec{r}$

$d\vec{r}$ can have any direction from P as long as it stays in the surface $\phi = C$

$\vec{\nabla}\phi$ is normal to the equipotential surface $\phi = C$.

Now, let $d\vec{r}$ take us from surface $\phi = C_1$ to another surface $\phi = C_2$:



Now, $d\phi = C_2 - C_1$

$= \vec{\nabla}\phi \cdot d\vec{r} \; (\neq 0) \; > 0$

$= |\vec{\nabla}\phi|\,|d\vec{r}|\cos\theta$

- For a given $d\phi$, $|d\vec{r}|$ minimum when $d\vec{r}$ chosen parallel to $\vec{\nabla}\phi$ ($\cos\theta = 1$ = maximum)
- For a given $|d\vec{r}|$, $d\phi$ maximum when $\vec{\nabla}\phi$ and $d\vec{r}$ are parallel

$\boxed{\Rightarrow \text{ identifies } \vec{\nabla}\phi \text{ as a vector having the direction of the maximum rate of change of } \phi}$

Possible Operations with $\vec{\nabla}$:

Divergence:

Let $\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

$$\vec{\nabla}\cdot\vec{V} \equiv \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(V_x \hat{x} + V_y \hat{y} + V_z \hat{z}\right)$$

$$= \underbrace{\partial_x V_x} + \partial_y V_y + \partial_z V_z \quad [\text{scalar}] = \overbrace{\frac{\partial}{\partial x_j} V_j}^{\text{summation}}$$

$$= \frac{\partial V_x}{\partial x} \quad (\text{shorthand})$$

Physical significance: "net flow /flux of a vector field" per unit volume

Curl: In index notation:

$$\vec{\nabla}\times\vec{V}: \quad (\vec{\nabla}\times\vec{V})_i = \sum_{j,k} \epsilon_{ijk}\frac{\partial}{\partial x_j} V_k = \overbrace{\underbrace{\epsilon_{ijk}}_{\text{sum over } j,k}\frac{\partial}{\partial x_j} V_k}^{\text{"implicit summation"}}$$

e.g., $(\vec{\nabla}\times\vec{V})_x = \epsilon_{xyz}\frac{\partial}{\partial y}V_z + \epsilon_{xzy}\frac{\partial}{\partial z}V_y$

$\left(\begin{array}{l} \text{"}1 \Leftrightarrow x\text{"} \\ 2 \Leftrightarrow y \\ 3 \Leftrightarrow z \end{array}\right) \quad = \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}$ ✓

[When a subscript appears twice in one side of an equation, Σ w.r.t. that subscript is implied]

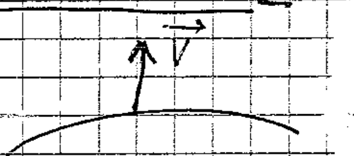
e.g., $\vec{A}\cdot\vec{B} = A_i B_i$

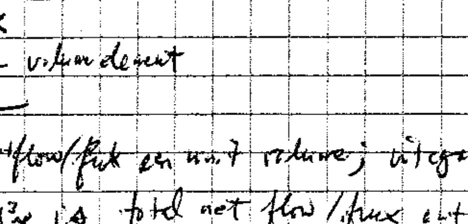
Physical significance: "rotation or circulation of a vector field"

Vector Integral Theorems:

for some vector field: $\vec{V}(x,y,z)$

Gauss's Theorem [(Divergence Theorem)]





(closed surface)

$$\underbrace{\oint_S \vec{V}\cdot \underset{\uparrow\ \text{surface element vector}}{d\vec{a}}} = \underbrace{\int_V (\vec{\nabla}\cdot\vec{V})\, \underset{\uparrow\ \text{volume element}}{d^3x}}$$

net flux through closed surface

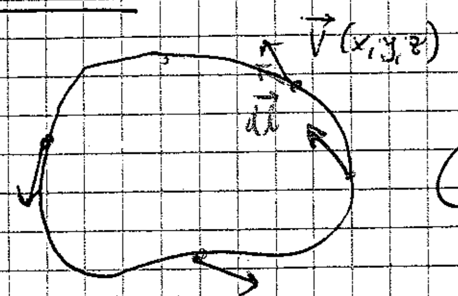
$\vec{\nabla}\cdot\vec{V}$: net outflow/flux per unit volume; integrate the entire volume

$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{V})\, d^3x$ is total net flow/flux out of the surface

equality via Gauss' Theorem

Divergence

Stokes' Theorem:



(open surface)

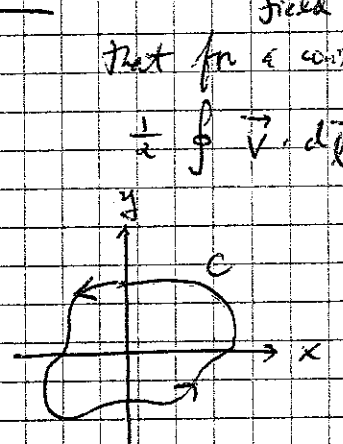
$$\underbrace{\oint_C \vec{V}\cdot d\vec{l}}_{\text{line integral around perimeter of the surface (closed path)}} = \underbrace{\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a}}_{\text{over surface bounded by the perimeter}}$$

below example illustrates directions of $d\vec{l}$ and $d\vec{a}$

$d\vec{a}$ direction right-hand rule!

Example: Given a vector field $\vec{V} = -y\hat{x} + x\hat{y}$, use Stokes' Theorem to show that for a continuous, closed curve in the xy-plane:

$$\frac{1}{2}\oint \vec{V}\cdot d\vec{l} = \text{area enclosed by curve}$$



Choose direction of traversal to be counter-clockwise,

$d\vec{a}$ by right-hand rule, out of paper $\Rightarrow$ $d\vec{a} = da\,\hat{z}$

$$\oint_C \vec{V}\cdot d\vec{l} = \oint_C \left[-y\hat{x} + x\hat{y}\right]\cdot\left[dx\,\hat{x} + dy\,\hat{y}\right]$$

$$= \oint_C \left[-y\,dx + x\,dy\right]$$

But how do we parametrize dx, dy for arbitrary C? (complicated!)

Use Stokes' Theorem:

$$\oint_C \vec{V}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a}$$

$$\vec{\nabla}\times\vec{V} = 2\hat{z}$$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = 2\cdot\int_S \hat{z}\cdot da\,\hat{z} = 2(\text{Area}) \checkmark$$

$$= 2\int_S da$$

Dirac Delta Function

Defined by its properties: (in one dimension)

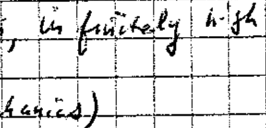
$$\begin{cases} \delta(x) = 0 & \text{for } x \neq 0 \\ f(0) = \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \end{cases} \xRightarrow{\text{special case}} \quad 1 = \int_{-\infty}^{\infty} \delta(x)\,dx \quad \text{for } f(x) = 1$$

- $\delta(x)$ can be intuitively thought of as an infinitely thin, infinitely high "spike" at $x = 0$
  e.g., point charge, or an impulsive force (mechanics)
- no such function exists in the usual mathematical definition of a function

Can be thought of intuitively (not rigorously!) as the limit of a Gaussian distribution.

recall: functional form: $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\dfrac{x^2}{2\sigma^2}\right]$

σ^2: variance



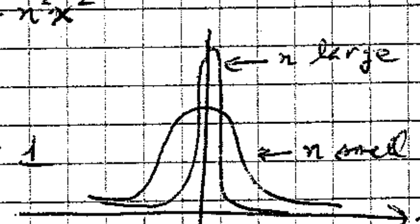
Define: $2\sigma^2 = \dfrac{1}{n^2} \Rightarrow \sigma^2 \to 0$ as $n \to \infty$

$$\Rightarrow f(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

$$\int_{-\infty}^{+\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}\,dx = 1$$

for all n

(normalized Gaussian distribution)
area remains constant



Rules:

$$\delta(ax) = \frac{1}{a}\,\delta(x) \quad \text{for } a > 0$$

$$\int_{-\infty}^{\infty} \delta'(x-a)\,dx = -f'(a) \qquad {}' \text{ denotes differentiation w.r.t. } x$$

$$\delta(g(x)) = \sum_{\substack{a,\\ \text{for } g(a)=0 \\ g'(a)\neq 0}} \frac{\delta(x-a)}{|g'(a)|} \qquad \text{(i.e., sum over the zeros of the function } g(x)\text{)}$$

To see first one:

$$\int_{-\infty}^{\infty} \delta(ax)\,dx \Rightarrow \begin{matrix} \text{let } y = ax \\ dy = a\,dx \end{matrix} \Rightarrow \int_{-\infty}^{\infty} \delta(y)\,\frac{dy}{a} = \frac{1}{a}\int_{-\infty}^{\infty} \delta(y)\,dy$$