Last time, considered image problem for point charge near grounded, conducting sphere.

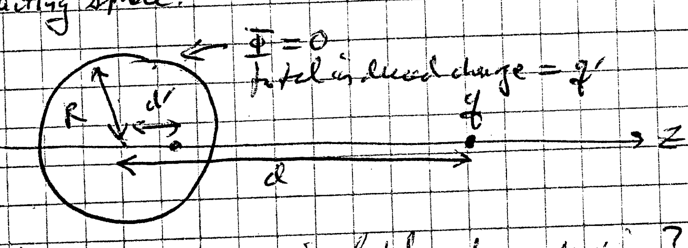


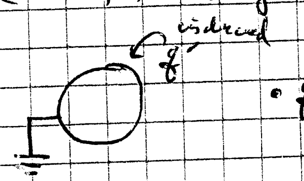
Image:

$$\left.\begin{aligned} q' &= -q\frac{R}{d} \\ d' &= \frac{R^2}{d} \end{aligned}\right\}$$

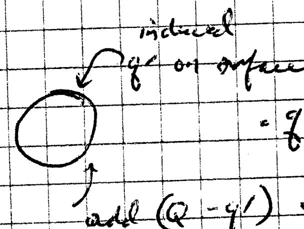
What if now the sphere is insulated and conducting? (with total charge Q)

Now, in presence of point charge q:

Appeal to superposition:



(1) Start with a grounded conducting sphere.
Charge q' will be distributed over its surface as we previously calculated.

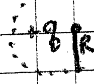


(2) Now, disconnect the ground wire.
Add charge $Q-q'$ to the sphere, so total charge on sphere is Q.

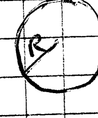
Note: the added charge $(Q-q')$ will distribute itself uniformly over the sphere, since the electrostatic forces due to q and q' are already balanced!

(3) From a viewpoint outside the sphere, the potential due to the added charge $(Q-q')$ will be the same as if a point charge of that magnitude is at the origin.

$$\left[\begin{array}{c}\text{just Gauss's} \\ \text{Law!}\end{array}\right]$$



$\Phi(r=R) = \dfrac{+q}{4\pi\epsilon_0 R}$



$r > R$: $\vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{r^2}\hat{r}$

$\Phi(r=R) = -\int_{\infty}^{R} \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{r^2}\,dr = \dfrac{q}{4\pi\epsilon_0 r}\Big|_{\infty}^{R}$ ✓

So for charge q near insulated, conducting sphere with charge Q:

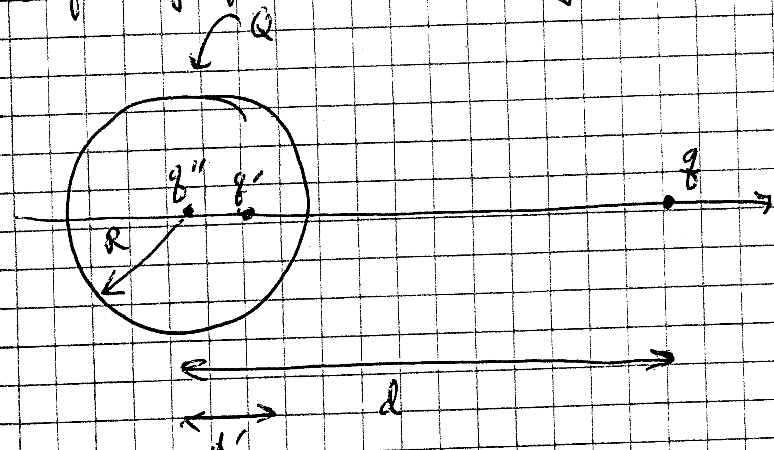
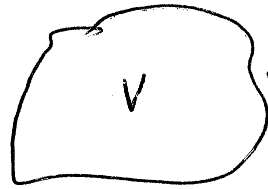


Image Charges (2)

$$\left[\begin{aligned} q' &= -q\frac{R}{d} \\ d' &= R^2/d \end{aligned}\right]$$

$$\left[\begin{array}{c} q'' = Q - q' \\ \text{at center of sphere} \end{array}\right]$$

Φ due to: q, q', and q'' for solving problems like this



← surface S

Back to Green's Theorem (3D):

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')\,d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G}{\partial n'}\right]da'$$

$\uparrow$

$\vec{x}$ in V

$$\vec{\nabla}'^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x},\vec{x}')$$

$$G(\vec{x},\vec{x}') = \underset{\substack{\uparrow \\ \text{potential of point charge}}}{\frac{1}{|\vec{x}-\vec{x}'|}} + \underbrace{F(\vec{x},\vec{x}')}_{\vec{\nabla}'^2 F(\vec{x},\vec{x}')=0 \text{ in } V}$$

"external (to V) potential" constructed to satisfy Dirichlet or Neumann boundary condition on $G(\vec{x},\vec{x}')$

Green Function for the Sphere

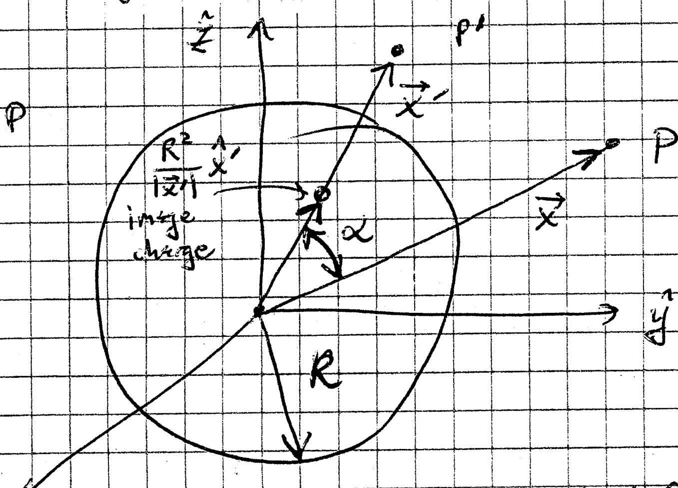
So far, given examples of method of images.

As mentioned during discussion of Green functions, (claimed)

* Potential due to a unit source and its image(s), chosen to satisfy (homogeneous boundary conditions), is just the Green function appropriate for Dirichlet or Neumann boundary conditions.

$G(\vec{x}, \vec{x}')$: appropriate choice for geometry in question allow impose b.c.'s (ρ, $\sigma_s \to 1$, σ_s in G's Thm) !!

- $\vec{x}$: point at which potential is being evaluated, P (fixed)
- $\vec{x}'$: integration variable for sources, P' (in volume V in Green's Theorem integrals)



"Exterior Problem"
- here V is volume exterior to the sphere, (i.e., $\vec{x} \in V$)
- image charge not in V!

Let: $\cos\alpha = \dfrac{\vec{x}\cdot\vec{x}'}{|\vec{x}|\cdot|\vec{x}'|}$

Construct G analogous to method of images

Image charge: (in units of q) $\quad -\dfrac{R}{|\vec{x}'|}$

at: $\dfrac{R^2}{|\vec{x}'|}\hat{x}' = \dfrac{R^2\vec{x}'}{|\vec{x}'|^2}$

Recall

$$G(\vec{x}, \vec{x}') = \underbrace{\frac{1}{|\vec{x}-\vec{x}'|}}_{\text{potential due to point charge}} + \underbrace{F(\vec{x}, \vec{x}')}_{\text{potential due to external system of charges chosen to satisfy b.c.'s on G}}$$

So, for our Dirichlet problem ($G_D = 0$ on sphere):

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \underbrace{-\frac{R}{|\vec{x}'|}\,\frac{1}{\left|\vec{x} - \frac{R^2}{|\vec{x}'|^2}\vec{x}'\right|}}_{F(\vec{x},\vec{x}')}$$

Green function for sphere (whether or not there are charges in Volume V)

$F(\vec{x}, \vec{x}')$: "image charge" used to construct $F(\vec{x}, \vec{x}')$ is outside V, so it will satisfy $\nabla'^2 F = 0$ inside V

$$= \frac{1}{\sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}} - \frac{R}{|\vec{x}'|}\frac{1}{\left[\left(\vec{x}-\frac{R^2}{|\vec{x}'|^2}\vec{x}'\right)\cdot\left(\vec{x}-\frac{R^2}{|\vec{x}'|^2}\vec{x}'\right)\right]^{1/2}}$$

[Let $x = |\vec{x}|$, $x' = |\vec{x}'|$]

$$= \frac{1}{\left[x^2 - 2xx'\cos\alpha + x'^2\right]^{1/2}} - \frac{R}{x'}\frac{1}{\left[x^2 - 2\frac{R^2}{x'^2}xx'\cos\alpha + \frac{R^4}{x'^4}x'^2\right]^{1/2}}$$

$$= \frac{1}{\left[x^2 + x'^2 - 2xx'\cos\alpha\right]^{1/2}} - \frac{1}{\left[\frac{x^2x'^2}{R^2} + R^2 - 2xx'\cos\alpha\right]^{1/2}}$$

So, now we have $G(\vec{x},\vec{x}')$!

Recall, for Dirichlet boundary conditions, $G_D(\vec{x},\vec{x}') = 0$ for $\vec{x}'$ on S (not $\vec{x}$!)

check if this is satisfied:

if $\vec{x}'$ is on S, $x' = R$:

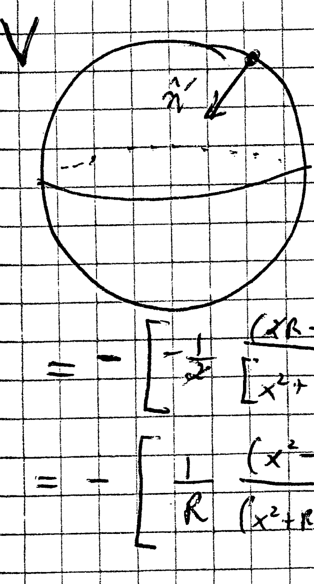
$$G = \frac{1}{\sqrt{x^2 + R^2 - 2xR\cos\alpha}} - \frac{1}{\sqrt{x^2 + R^2 - 2xR\cos\alpha}} = 0 \quad \checkmark$$

could brute-force check that: $\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$ but not so enlightening

General solution for Dirichlet case is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da'$$

Now, for our exterior problem:



V is everywhere exterior to the sphere.

$\hat{n}'$ is radially inwards! (anti-parallel to $\vec{x}'$)

$$\Rightarrow \frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial x'} \qquad (|\vec{x}'| = r')$$

$$= -\left[-\frac{1}{2}\frac{(2R - 2x\cos\alpha)}{\left[x^2+R^2-2xR\cos\alpha\right]^{3/2}} + \frac{1}{2}\frac{\left(\frac{2x^2}{R} - 2x\cos\alpha\right)}{\left[x^2+R^2-2xR\cos\alpha\right]^{3/2}}\right]$$

$$= -\left[\frac{1}{R}\frac{(x^2 - R^2)}{(x^2+R^2-2xR\cos\alpha)^{3/2}}\right]$$