So, the solution for $\Phi(\vec{x})$ will be of the form:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\iint_S \Phi(R,\theta',\phi')\cdot R\frac{(x^2-R^2)}{(x^2+R^2-2xR\cos\alpha)^{3/2}} \sin\theta'\, d\theta'\, d\phi'$$

Suppose we consider the image problem from before, where:

(q near sphere, $\Phi = 0$ on sphere)

$$\Phi_S(R,\theta';\phi') = 0$$

$\Rightarrow$ so have charge distribution now

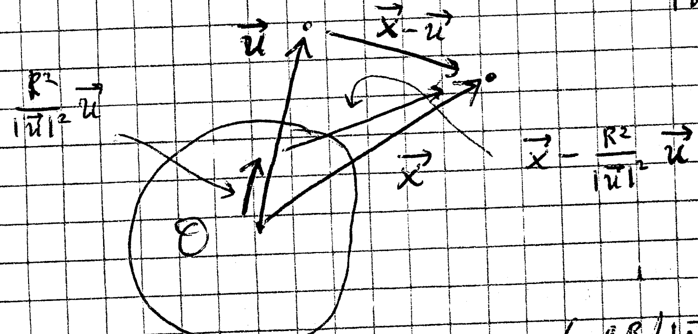
$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3x'$$

[if $\rho(\vec{x}') = q\cdot\delta(\vec{x}'-\vec{u})$, point charge at $\vec{u}$]

$$= \frac{1}{4\pi\varepsilon_0}\int_V q\cdot\delta(\vec{x}'-\vec{u})\, G_D(\vec{x},\vec{x}')\, d^3x'$$

$$= \frac{q}{4\pi\varepsilon_0} G_D(\vec{x},\vec{u})$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{u}|} - \frac{qR}{4\pi\varepsilon_0|\vec{u}|}\cdot\frac{1}{\left|\vec{x}-\frac{R^2}{|\vec{u}|^2}\vec{u}\right|}$$



$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{u}|} + \frac{(-qR/|\vec{u}|)}{4\pi\varepsilon_0}\cdot\frac{1}{\left|\vec{x}-\frac{R^2}{|\vec{u}|^2}\vec{u}\right|} \quad \checkmark$$

For $\Phi(R,\theta';\phi') \neq 0$ on sphere surface, not so easy!!

Need to do surface integral:

$$\frac{1}{4\pi}\iint_S \Phi(R,\theta',\phi')\cdot R\frac{x^2-R^2}{(x^2+R^2-2xR\cos\alpha)^{3/2}}\sin\theta'\, d\theta'\, d\phi'$$

$\cos\alpha$ = angle between $\vec{x}$ and $\vec{x}'$

$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

(θ,ϕ) for $\vec{x}$, (θ',ϕ') for $\vec{x}'$

$$\text{if: } \vec{x} = \begin{cases} x \sin\theta \cos\phi \\ x \sin\theta \sin\phi \\ x \cos\theta \end{cases} \qquad \vec{x}' = \begin{cases} x' \sin\theta' \cos\phi' \\ x' \sin\theta' \sin\phi' \\ x' \cos\theta' \end{cases}$$

$$\cos\alpha = \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}||\vec{x}'|} = \frac{xx' \sin\theta\cos\phi \sin\theta'\cos\phi' + xx'\sin\theta\sin\phi\sin\theta'\sin\phi' + xx'\cos\theta\cos\theta'}{xx'}$$

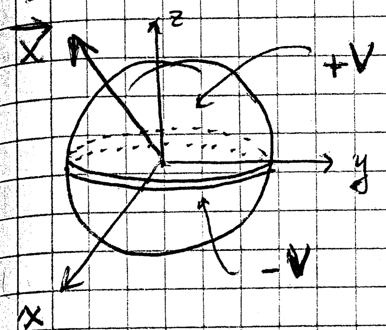
$$= \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi' + \cos\theta\cos\theta'$$

$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\left[\cos\phi\cos\phi' + \sin\phi\sin\phi'\right]$$

$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \quad \checkmark$$

$$\left[\text{as } \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\right]$$

Another application of our Green function for a sphere:



(no charges in volume V)

$$\Rightarrow \rho(\vec{x}\,') = 0 \quad \text{so} \quad \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}\,')\, G_D(\vec{x},\vec{x}\,') = 0$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi}\oint_S \Phi(R,\theta',\phi')\cdot \overbrace{R\,\frac{(x^2-R^2)}{(x^2+R^2-2xR\cos\alpha)^{3/2}}}^{=\,\partial G/\partial n'|_S} \sin\theta'\, d\theta'\, d\phi'$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi/2} \sin\theta'\, d\theta'\, (+V)\, \frac{R(x^2-R^2)}{(x^2+R^2-2xR\cos\alpha)^{3/2}}$$

$$+ \frac{1}{4\pi}\int_0^{2\pi} d\phi' \int_{\pi/2}^{\pi} \sin\theta'\, d\theta'\, (-V)\, \frac{R(x^2-R^2)}{(x^2+R^2-2xR\cos\alpha)^{3/2}}$$

As $\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

generally cannot do integrals in analytic form!

Special case: $\Phi(\vec{x})$ along z-axis; $\vec{x} \parallel \hat{z} \Rightarrow \cos\theta = 1$, $|\vec{x}| = z$

$\cos\alpha = \cos\theta'$

$$\Rightarrow \Phi(\vec{x}) = \frac{V}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi/2} \sin\theta'\, d\theta'\, \frac{R(x^2-R^2)}{(x^2+R^2-2xR\cos\theta')^{3/2}}$$

$$- \frac{V}{4\pi}\int_0^{2\pi} d\phi' \int_{\pi/2}^{\pi} \frac{R(x^2-R^2)}{(x^2+R^2-2xR\cos\theta')^{3/2}}\sin\theta'\, d\theta'$$

$$= -\frac{V}{4\pi}(R)(x^2-R^2)(2\pi)(x^2+R^2-2xR\cos\theta')^{-1/2}\cdot\frac{1}{xR}\Bigg|_0^{\pi/2}$$

$$+ \frac{V}{4\pi}R(x^2-R^2)(2\pi)(x^2+R^2-2xR\cos\theta')^{-1/2}\frac{1}{xR}\Bigg|_{\pi/2}^{\pi}$$

$$= -\frac{V}{2}(x^2-R^2)\frac{1}{x}\left[\frac{1}{\sqrt{x^2+R^2}} - \frac{1}{\sqrt{x^2+R^2-2xR}}\right]$$

$$+ \frac{V}{2}(x^2-R^2)\frac{1}{x}\left[\frac{1}{\sqrt{x^2+R^2+2xR}} - \frac{1}{\sqrt{x^2+R^2}}\right]$$

$$= V\left[1 - \frac{x^2-R^2}{x\sqrt{x^2+R^2}}\right]$$

Orthogonal Functions and Expansions

So far, two ways to solve boundary-value problems (i.e., Poisson equation):
- Green function
- Method of images

Next way: Separation of variables for direct solution to the partial differential equation $\vec{\nabla}^2 \Phi = 0$

But first: Orthogonal functions

Consider interval (a, b) in variable ξ ("xi") with set of real or complex functions $U_n(\xi)$, $n = 1, 2, \ldots$, square integrable, and orthogonal on (a, b).

Recall: square integrable: $\int_a^b |U_n(\xi)|^2 d\xi < \infty$ [recall: if $U_n(\xi)$ complex-valued $|U_n(\xi)|^2 = U_n^*(\xi) \cdot U_n(\xi)$]

orthogonality: $\int_a^b U_n^*(\xi) U_m(\xi) \, d\xi = 0$ for $m \neq n$, non-zero for $m = n$

If functions normalized so that integral is 1 for $m = n$, "orthonormal"

$$\int_a^b U_n^*(\xi) U_m(\xi) \, d\xi = \delta_{mn}$$

Arbitrary function $f(\xi)$ (square integrable) can be expanded in a series of the orthonormal $U_n(\xi)$:

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) = a_1 U_1(\xi) + a_2 U_2(\xi) + \ldots$$

What are the a_n?

write: $U_m^*(\xi) f(\xi) = U_m^*(\xi) \left[\sum_{n=1}^{\infty} a_n U_n(\xi) \right]$

integrate both sides w.r.t. $\int_a^b d\xi$:

$$\int_a^b d\xi \, U_m^* f(\xi) = \int_a^b d\xi \, U_m^*(\xi) \left[\sum_{n=1}^{\infty} a_n U_n(\xi) \right]$$

using: $\int_a^b U_m^*(\xi) U_n(\xi) \, d\xi = \delta_{mn}$

$$\boxed{\int_a^b U_m^*(\xi) f(\xi) \, d\xi = a_m}$$

[not possible to define if U_n not square-integrable]

So explicitly we now have:

ξ': dummy variable

$$f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) = \sum_{n=1}^{\infty} \left[\int_a^b U_n^*(\xi') f(\xi') \, d\xi' \right] U_n(\xi)$$

$$= \int_a^b \left[\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) \right] f(\xi') \, d\xi'$$

see next page

$$\sum_{n=1}^{\infty} \left[\int_a^b u_n^*(\xi') f(\xi') d\xi' \right] u_n(\xi)$$

$$= \left[\int_a^b u_1^{*(\xi')} f^{(\xi')} d\xi' \right] \cdot u_1 + \left[\int_a^b u_2^{*(\xi')} f^{(\xi')} d\xi' \right] \cdot u_2 + \ldots\ldots$$

$$= \int_a^b d\xi' \left[u_1(\xi) u_1^*(\xi') f(\xi') + u_2(\xi) u_2^*(\xi') f(\xi') + \ldots \right]$$

$$= \int_a^b d\xi' \left[\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) \right] f(\xi') \quad \checkmark$$

Now, this represents any function on the interval $[a,b]$. So we must have: (11)

$$f(\xi) = \int_a^b \left[\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) \right] f(\xi')\, d\xi'$$

Recall: $g(b) = \int_a^c g(x')\, \delta(x'-b)\, dx'$ if $a<b<c$

for arbitrary $g(x')$

From this, we identify:

$$\sum_{n=1}^{\infty} u_n^*(\xi') u_n(\xi) = \delta(\xi'-\xi)$$ } completeness or closure relation

<u>Fourier Series</u>: variable x (real-valued), consider interval $[-a/2, +a/2]$

Orthonormal functions are: $\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right)$; $\sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)$

$m \geq 1$ $\qquad$ $m \geq 1$

$m=0$: $\frac{1}{\sqrt{a}}$ for cosine ✓

Can write:

$\Rightarrow f(x) = \sum a_n u_n$; $\left[a_n = \int_a^b u_n^*(\xi') f(\xi') d\xi' \right]$

$$= \underbrace{\int_{-a/2}^{a/2} \left[\frac{1}{\sqrt{a}} f(x)\,dx\right] \frac{1}{\sqrt{a}}}_{m=0 \text{ of cos expansion}} + \sum_{m=1}^{\infty} \left[\sqrt{\frac{2}{a}} \int_{-a/2}^{a/2} \cos\left(\frac{2\pi m x}{a}\right) f(x)\,dx \right] \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi m x}{a}\right)$$

$$+ \sum_{m=1}^{\infty} \left[\sqrt{\frac{2}{a}} \int_{-a/2}^{a/2} \sin\left(\frac{2\pi m x}{a}\right) f(x)\,dx \right] \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi m x}{a}\right)$$

$$\equiv \frac{1}{2} A_0 + \sum_{m=1}^{\infty} \left[A_m \cos\left(\frac{2\pi m x}{a}\right) + B_m \sin\left(\frac{2\pi m x}{a}\right) \right]$$

$$A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos\left(\frac{2\pi m x}{a}\right) dx \; ; \quad B_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin\left(\frac{2\pi m x}{a}\right) dx$$

$m \geq 0$ $\qquad$ $m \geq 1$