7b

Now suppose (a, b) becomes infinite, $(-\infty, \infty)$. For example, consider the Fourier integral.

Start with: (orthonormal set of complex exponentials)

$$\left.\begin{aligned} u_m(x) &= \frac{1}{\sqrt{a}} \exp\left[i\,(2\pi m x/a)\right] \\ m &= 0, \pm 1, \pm 2, \ldots \quad \text{on } (-a/2,\ a/2) \end{aligned}\right\}$$

The expansion of $f(x)$ then becomes; $f(x) = \Sigma\, a_n u_n$

$$f(x) = \sum_{m=-\infty}^{\infty} \left[\int_{-a/2}^{a/2} \frac{1}{\sqrt{a}}\, e^{-i(2\pi m x'/a)} f(x')\, dx' \right] e^{i(2\pi m x/a)} \frac{1}{\sqrt{a}}$$

$$= \sum_{m=-\infty}^{\infty} A_m \frac{e^{i(2\pi m x/a)}}{\sqrt{a}}$$

$$A_m = [\ \ldots\ ]$$

Now let $a \to \infty$, so interval becomes infinite $(-\infty, \infty)$. Also transform:

$$k \equiv \frac{2\pi m}{a} \quad \left(\Rightarrow \quad dk = \frac{2\pi}{a}\, dm \right)$$

$$\sum_m \to \int dm = \frac{a}{2\pi} \int dk$$

$$\Rightarrow \quad f(x) = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}}\, e^{-i(kx')} f(x')\, dx' \right] e^{ikx} \frac{1}{\sqrt{a}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik\cdot x} \left[\int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx' \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx' \right]}_{\equiv A(k)}$$

$$\left.\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} A(k) \right\} \text{"Fourier Integral"}$$

Note: ~~Since must have for arbitrary $f(x)$:~~

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx'\, e^{ik(x-x')} f(x') \Rightarrow \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = 2\pi\, \delta(x-x')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') \cdot \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}$$

Consider $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k)\, e^{ikx}$ (43)

(view/interpret as):

$$= \int_{-\infty}^{\infty} dk\, \underbrace{A(k)}_{\text{coefficients}}\, \underbrace{\frac{e^{ikx}}{\sqrt{2\pi}}}_{\text{"expansion functions" in } k} \Bigg\} \text{ "expansion" in } k \quad \int dk \leftrightarrow \sum_k$$

$$\left[\begin{array}{l} \text{Orthogonality requires:} \\ \int U_n^*(\xi)\, U_m(\xi)\, d\xi = 0 \quad m \neq n \\ = 1; \ m = n, \end{array} \right] \int_{-\infty}^{\infty} \frac{e^{-ik'x}}{\sqrt{2\pi}} \cdot \frac{e^{ikx}}{\sqrt{2\pi}}\, dx = \begin{cases} 1 & \text{if } k' = k \\ 0 & \text{else} \end{cases}$$

$$\rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k-k')$$

Separation of Variables

Recall Laplace Equation is: $\vec{\nabla}^2 \Phi = 0$

In rectangular coordinates: $\partial_x^2 \Phi + \partial_y^2 \Phi + \partial_z^2 \Phi = 0$

Under the assumption: $\Phi(x,y,z) = f(x)\, g(y)\, h(z)$

$$\left(\frac{d^2 f}{dx^2}\right) g(y)\, h(z) + \left(\frac{d^2 g}{dy^2}\right) f(x)\, h(z) + \left(\frac{d^2 h}{dz^2}\right) f(x)\, g(y) = 0$$

$$\underbrace{\frac{1}{f(x)} \frac{d^2 f}{dx^2}}_{\text{depends only on } x} + \underbrace{\frac{1}{g(y)} \frac{d^2 g}{dy^2}}_{\ldots y} + \underbrace{\frac{1}{h(z)} \frac{d^2 h}{dz^2}}_{\ldots z} = 0$$

This must hold no matter where you are inside some volume with no charge, i.e., must be independent of x, y, and z, so must have: [e.g., if (y,z) fixed, vary x]

$$\left. \begin{array}{l} \dfrac{1}{f(x)} \dfrac{d^2 f}{dx^2} \equiv -\alpha^2 \quad \text{(constant)} \\[2ex] \dfrac{1}{g(y)} \dfrac{d^2 g}{dy^2} \equiv -\beta^2 \\[2ex] \dfrac{1}{h(z)} \dfrac{d^2 h}{dz^2} = \gamma^2 \end{array} \right\} \quad \alpha^2 + \beta^2 = \gamma^2$$

Now, choose (arbitrarily) $\alpha^2 > 0$ and $\beta^2 > 0$.

Easy now to solve these three independent differential equations:

$$\frac{1}{f}\frac{d^2 f}{dx^2} = -\alpha^2 \Rightarrow f = e^{\pm i\alpha x}, \ \sin(\alpha x), \cos(\alpha x)$$

$$g = e^{\pm i\beta y}, \ \sin(\beta y), \cos(\beta y)$$

$$h = e^{\pm \gamma z} = e^{\pm\sqrt{\alpha^2+\beta^2}\, z}, \ \sinh\left[\sqrt{\alpha^2+\beta^2}\, z\right], \ \cosh\left[\sqrt{\alpha^2+\beta^2}\, z\right]$$

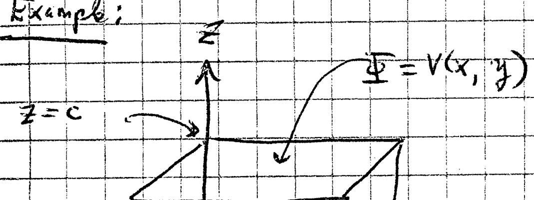
a linear combinations thereof!

e.g., possible solution:

$$\Rightarrow \Phi(x,y,z) = e^{\pm i\alpha x} e^{\pm i\beta y} e^{\pm\sqrt{\alpha^2+\beta^2}\, z}$$

To find the α and β need to impose specific boundary conditions.

__Example:__



Suppose all surfaces of box at $\Phi = 0$ except for top face, at $\Phi = V(x,y)$

Need $\Phi = 0$ at $x=0, y=0, z=0$

$$\Rightarrow f = \sin\alpha x$$

$$g = \sin\beta y$$

$$h = \sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right) \longrightarrow$$

To have $\Phi = 0$ at $x=a$ and $y=b$:

$$\left.\begin{array}{l} \alpha_n = \frac{n\pi}{a} \text{ will work} \\ \beta_n = \frac{m\pi}{b} \text{ will work} \end{array}\right\} \Rightarrow \gamma_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b}} = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$

$$\Rightarrow \Phi_{nm} = \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)\sinh\left(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\, z\right)$$

$$\equiv \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

$$\Phi(x,y,z) = \underbrace{\sum_{n,m=1}^{\infty}}_{\text{linear superposition of all possible solutions}} \underbrace{A_{nm}}_{\text{expansion coefficients}} \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$