Final boundary condition at $z = c$:

$$V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x)\, \sin(\beta_m y)\, \overbrace{\sinh(\gamma_{nm} c)}^{\text{constant}}$$

This is just a two-dimensional (discrete) Fourier series!
up to the "$\sinh(\gamma_{nm} c)$ constant"

$$A_{nm} = \frac{4}{ab}\,\frac{1}{\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\, V(x,y) \sin(\alpha_n x) \times \sin(\beta_m y)$$

Now, consider a 2-D (e.g., if Φ is indep't of one coordinate) potential problem we can solve by a Fourier series technique.

In 2-D, Laplace equation is: $\dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial y^2} = 0$

Assuming: $\Phi(x,y) = f(x)\cdot g(y) \Rightarrow$

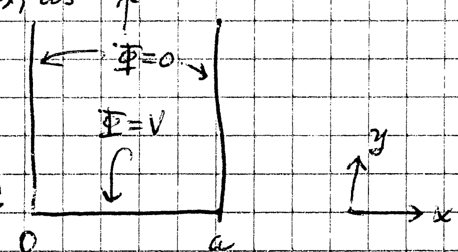
$$\frac{d^2 f}{dx^2}\, g(y) + \frac{\partial^2 g}{\partial y^2}\, f(x) = 0$$

$$\underbrace{\frac{d^2 f}{dx^2}\cdot\frac{1}{f(x)}}_{\equiv -\alpha^2} + \underbrace{\frac{d^2 g}{dy^2}\,\frac{1}{g(y)}}_{\equiv +\alpha^2} = 0$$

$$\frac{d^2 f}{dx^2} = -\alpha^2 f(x) \qquad \frac{d^2 g}{dy^2} = \alpha^2 g(y)$$

$$f(x) = \underbrace{e^{\pm i\alpha x}}_{\sin(\alpha x),\ \cos(\alpha x)} \qquad g(y) = e^{\pm \alpha y} \Rightarrow \Phi(x,y) \propto \underbrace{e^{\pm i\alpha x}}_{\substack{\sin(\alpha x)\\ \cos(\alpha x)}} e^{\pm \alpha y}$$

Apply this to:
want potential
Φ in $0 < x < a$,
$y \geq 0$ region!



- Need $\Phi = 0$ at $x = 0, a \Rightarrow \sin\left(\dfrac{n\pi x}{a}\right) \qquad \left[\alpha = \dfrac{n\pi}{a}\right]$
- Need $\Phi \to 0$ for $y \to \infty \Rightarrow e^{-n\pi y/a}$

Solution is linear combination:

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \exp(-n\pi y/a) \sin\left(\frac{n\pi x}{a}\right)$$

boundary condition is:

$$V = \Phi(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \longrightarrow \text{just a Fourier series!}$$
$[0 \to a]$ vs. $\left[-\frac{a}{2}, +\frac{a}{2}\right]$

$$A_n = \frac{2}{a}\int_0^a \Phi(x,0) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2V}{a}\int_0^a \sin(n\pi x/a)\, dx = \frac{2V}{a}\cdot\frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\Bigg|_a^0 =$$

$$V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c) \cdot \sin(\alpha_n x) \cdot \sin(\beta_m y)$$

$$\Rightarrow \int_0^a dx \sin(\alpha_i x) \int_0^b dy \sin(\beta_j y) \cdot V(x,y) =$$

$$= \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c) \cdot \int_0^a dx \sin(\alpha_n x) \sin(\alpha_i x) \cdot \int_0^b dy \sin(\beta_m y) \sin(\beta_j y) \cdot \qquad \left(\begin{aligned} \alpha_\ell &= \frac{\ell\pi}{a} \\ \beta_\ell &= \frac{\ell\pi}{b} \end{aligned} \right)$$

$$\int_0^a dx \sin(\alpha_n x) \sin(\alpha_i x) = \left. \frac{\sin[(\alpha_n - \alpha_i) x]}{2(\alpha_n - \alpha_i)} \right|_0^a - \left. \frac{\sin[(\alpha_n + \alpha_i) x]}{2(\alpha_n + \alpha_i)} \right|_0^a \quad [\text{for } \alpha_n \neq \alpha_i]$$

$$= \left. \frac{\sin\left[\frac{(n-i)\pi}{a} x\right]}{2\frac{(n-i)\pi}{a}} \right|_0^a - \left. \frac{\sin\left[\frac{(n+i)\pi}{a} x\right]}{2\frac{(n+i)\pi}{a}} \right|_0^a = 0 \quad \underline{\underline{\text{for } i \neq n}}$$

as $\sin q\pi = 0 \ \forall q$ integers

<u>if $\alpha_n = \alpha_i$:</u>

$$\int_0^a dx \sin(\alpha_n x) \sin(\alpha_n x) = \int_0^a dx \sin^2(\alpha_n x) = \left[\frac{x}{2} - \frac{1}{2\alpha_n} \sin \alpha_n x \cos \alpha_n x \right]_0^a$$

$$= \frac{a}{2}$$

$$\Rightarrow \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_i x) \sin(\beta_j y) = A_{ij} \sinh(\gamma_{ij} c) \cdot \frac{a}{2} \cdot \frac{b}{2}$$

$$\Rightarrow \boxed{A_{ij} = \frac{4}{ab} \cdot \frac{1}{\sinh(\gamma_{ij} c)} \cdot \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_i x) \sin(\beta_j y)}$$

$$= \frac{2V}{n\pi}\cdot\left[1-\cos(n\pi)\right] = \frac{2V}{n\pi}\cdot\begin{cases}0 & n \text{ even}\\ 2 & n \text{ odd}\end{cases}$$

$$\Rightarrow \Phi(x,y) = \sum_{n\ \text{odd}}\left(\frac{4V}{n\pi}\right)e^{-n\pi y/a}\sin\left(\frac{n\pi x}{a}\right)$$

We can sum this series:

Recall: $\sin\alpha = \mathrm{Im}\left(e^{i\alpha}\right)$

$$\Rightarrow \Phi(x,y) = \frac{4V}{\pi}\,\mathrm{Im}\left\{\sum_{n\ \text{odd}}\frac{1}{n}\cdot e^{-n\pi y/a}\,e^{in\pi x/a}\right\}$$

$$= \frac{4V}{\pi}\,\mathrm{Im}\left\{\sum_{n\ \text{odd}}\frac{1}{n}\,e^{i\cdot i n\pi y/a}\,e^{in\pi x/a}\right\}$$

$$= \frac{4V}{\pi}\,\mathrm{Im}\left\{\sum_{n\ \text{odd}}\frac{1}{n}\,e^{(in\pi/a)(iy+x)}\right\}$$

Let: $Z = e^{(i\pi/a)(iy+x)}$

$$\Rightarrow \Phi(x,y) = \frac{4V}{\pi}\,\mathrm{Im}\left\{\sum_{n\ \text{odd}}\frac{Z^n}{n}\right\}$$

Using: $\ln(1+Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \frac{1}{4}Z^4 + \ldots$ (around $z=0$) Valid for complex numbers

$(Z\to -Z)$ $\Rightarrow \ln(1-Z) = -Z - \frac{1}{2}Z^2 - \frac{1}{3}Z^3 - \frac{1}{4}Z^4 + \ldots$

$$\Rightarrow \ln(1+Z) - \ln(1-Z) = \left(Z - \tfrac{1}{2}Z^2 + \tfrac{1}{3}Z^3 - \tfrac{1}{4}Z^4 + \ldots\right) - \left(-Z - \tfrac{1}{2}Z^2 - \tfrac{1}{3}Z^3 - \tfrac{1}{4}Z^4 + \ldots\right)$$

$$= 2\left(Z + \tfrac{1}{3}Z^3 + \ldots\right)$$

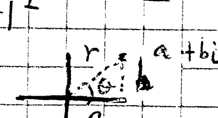
$$\Rightarrow \Phi(x,y) = \frac{2V}{\pi}\,\mathrm{Im}\left[\ln(1+Z) - \ln(1-Z)\right]$$

$$= \frac{2V}{\pi}\,\mathrm{Im}\left[\ln\left(\frac{1+Z}{1-Z}\right)\right]$$

for some $Z = a+bi$

$Z - Z^* = (a+bi)-(a-bi)$
$= 2bi$
$= 2i\,\mathrm{Im}\,Z$

Simplify: $\left(\frac{1+Z}{1-Z}\right) = \frac{(1+Z)(1-Z)^*}{(1-Z)(1-Z)^*} = \frac{1+Z-Z^*-|Z|^2}{|1-Z|^2} = \frac{1+2i\,\mathrm{Im}\,Z - |Z|^2}{|1-Z|^2}$



In general: $\mathrm{Im}\left[\ln(a+bi)\right]$

$$= \mathrm{Im}\left[\ln\left(\underbrace{\sqrt{a^2+b^2}}\,\underbrace{e^{i\tan^{-1}(b/a)}}\right)\right] = \mathrm{Im}\left[\ln\sqrt{a^2+b^2} + \ln e^{i\tan^{-1}\left(\frac{b}{a}\right)}\right]$$

$$= \tan^{-1}\left(\frac{b}{a}\right)$$

So we see:

$$\Phi(x,y) = \frac{2V}{\pi}\,\mathrm{Im}\left[\ln\left(\frac{1-|z|^2+i\,2\,\mathrm{Im}\,z}{|1-z|^2}\right)\right]$$

$$= \frac{2V}{\pi}\tan^{-1}\left(\frac{2\,\mathrm{Im}\,z}{1-|z|^2}\right)$$

Restoring $z = e^{(i\pi/a)(x+iy)} = e^{(i\pi x/a\, -\, \pi y/a)} = \left[\cos\left(\frac{\pi x}{a}\right)+i\sin\left(\frac{\pi x}{a}\right)\right]\cdot e^{-\pi y/a}$

$$\Phi(x,y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\cdot\sin\left(\frac{\pi x}{a}\right)e^{-\pi y/a}}{1-e^{-2\pi y/a}}\right)$$

$|z|^2 = zz^* = e^{-2\pi y/a}$

$$= \frac{2V}{\pi}\tan^{-1}\left(\frac{2\sin\left(\frac{\pi x}{a}\right)}{e^{\pi y/a}-e^{-\pi y/a}}\right)$$

$$= \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin\left(\frac{\pi x}{a}\right)}{\sinh\left(\frac{\pi y}{a}\right)}\right)$$

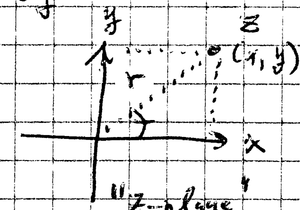
$\sinh x = \frac{1}{2}\left(e^{x}-e^{-x}\right)$

~~Solutions to Laplace Equation in 2-D via~~ Conformal Mapping (not discussed in Jackson)

See, e.g., "Fundamentals of Complex Analysis with Applications to Engineering, Science and Mathematics", Saff & Snider

Recall: complex variable $z = x + iy$

Representation of z in the xy-plane:



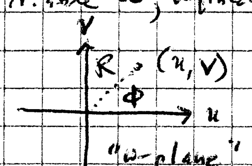
$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$z = re^{i\theta}$

Suppose there is a different complex variable w, defined by:

$$w = u + iv = Re^{i\phi}$$



$R = \sqrt{u^2+v^2}$

$\phi = \tan^{-1}\left(\frac{v}{u}\right)$

$w = Re^{i\phi}$

If w is a function of z, such that for each value of z there is some rule which specifies the corresponding value of w, we have:

$$w = f(z) = u(x,y) + i\,v(x,y)$$

Derivative of $w = f(z)$ defined in the usual way:

$$\frac{dw}{dz} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$