Note, however, that Δz may (in principle) have any direction, as it could result from some change in the x- and/or y-direction(s), as $\Delta Z = \Delta X + i\Delta y$, approach $\Delta z \to 0$ any direction in complex plane.

If $\frac{dw}{dz}$ has a unique value at z, regardless of path, $w(z)$ said to be "analytic" at z.

Now, if derivative independent of path, same result for $\frac{dw}{dz}$ must be obtained if z changed solely in x-direction, or solely in y-direction.

Since: $w = u(x,y) + iv(x,y)$

$$\left.\frac{dw}{dz}\right|_{\text{"}\Delta x\text{ only"}} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\,\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Similarly,

$$\left.\frac{dw}{dz}\right|_{\text{"}\Delta y\text{ only"}} = \lim_{i\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{\underbrace{i\Delta y}_{\text{the change in } z \text{ for a } \Delta y!}}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

If w is to be analytic at z, then require:

$(*)\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (equating Re parts)

$(**)\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ (Im parts)

"Cauchy–Riemann Equations"

[Note: since any Δz can be expressed as linear sum of $\Delta x + i\Delta y$, the C-R equations sufficient condition for analyticity.]

Trivial Example: $w = z^2 \Rightarrow u + iv = (x+iy)(x+iy) = x^2 + i2xy - y^2$

$u = x^2 - y^2$

$v = 2xy$

$\partial_x u = 2x \qquad \partial_y v = 2x \quad \checkmark$

$\partial_x v = 2y \qquad -\partial_y u = 2y \quad \checkmark$

Differentiate (*) w.r.t. x and (**) w.r.t. y:

$$(*) \rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

$$(**) \rightarrow \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

(sum)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly, diff' (*) w.r.t. y, (**) w.r.t. x:

$$(*) \rightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \Rightarrow -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$(**) \rightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ Both the Real & Imaginary Parts of an analytic function of a complex variable $w = f(z)$ satisfy the Laplace equation in 2-D !!

$\Rightarrow$ Theory of complex functions rich application to electrostatic potential problems.

Interesting Note: if one of two components of w is chosen to represent Φ, the other component is related to the electric field

$[w = u(x,y) + iv(x,y)]$

Let: $u(x,y)$ be chosen as the potential function for a particular problem.

As $\vec{E} = -\vec{\nabla} u \Rightarrow E_x = -\frac{\partial u}{\partial x} \quad E_y = -\frac{\partial u}{\partial y}$

Recall: $\nabla u = \partial_x u \, \hat{x} + \partial_y u \, \hat{y}$ is $\perp$ to $u(x,y) =$ constant (equipotentials)
tangent to flux line ($\vec{E}$)

But: $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ (by C-R equations)

In moving along a contour $v =$ constant, $dv = 0 \Rightarrow \frac{dx}{dy} = \frac{\partial u/\partial x}{\partial u/\partial y}$ } ratio of components of vector $\parallel$ to $\partial_x u \hat{x} + \partial_y u \hat{y}$

$\Rightarrow$ contours $v =$ constant coincide with $\vec{E}$-field lines (not magnitude of $\vec{E}$! which follows from $\oint \vec{E} \cdot d\vec{l}$)

(optional, if time)

Further, we have: (in general)

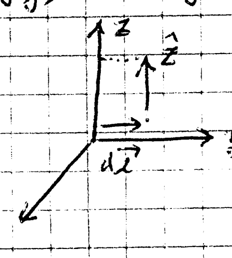
$$dv = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

$$= E_y\,dx - E_x\,dy$$

If we define $\vec{dl} \equiv dx\,\hat{x} + dy\,\hat{y}$, we can define a surface element $d\vec{S}$:

$$d\vec{S} \equiv \vec{dl} \times \hat{z} = \underbrace{-dx\,\hat{y} + dy\,\hat{x}}$$

a rectangle, side dl (in the xy-plane), and side 1 in the z-direction [e.g., if $\vec{dl} = dy\,\hat{y}$, $d\vec{S} = dy\,\hat{x}$]



correct by right-hand rule!

Electric flux through this surface element is: (if $\Psi \equiv \int \vec{E}\cdot d\vec{a}$)

$$d\Psi = \vec{E}\cdot d\vec{S} = (E_x\,\hat{x} + E_y\,\hat{y})\cdot(dy\,\hat{x} - dx\,\hat{y})$$

$$= E_x\,dy - E_y\,dx$$

$$\Rightarrow d\Psi = -dv \Rightarrow \Psi = -v + \text{constant}$$

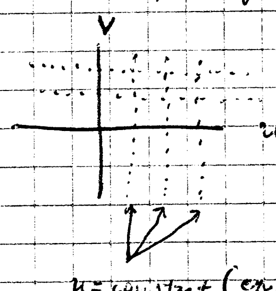
$\Rightarrow$ total electric flux between two contours $v = v_1$ and $v = v_2$, per unit length in the z-direction, is $-(v_2 - v_1)$.

∴ contours of v not only the field lines themselves, but v can be made a measure of the total Electric flux

If: • $u(x,y)$ = constant equipotentials

• $v(x,y)$ = constant, $\vec{E}$-field lines

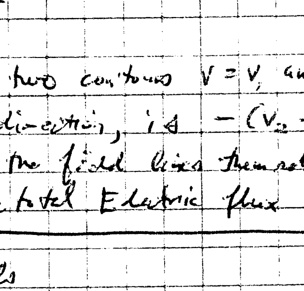
In w-plane,



v = constant ($\vec{E}$-field lines)

u = constant (equipotentials)

This is just the electrostatic field picture for: region between parallel conducting plates, $\pm\sigma$, in w-space. (uv-plane)



constant potential $V = -\int_{x_1}^{x} \vec{E}\cdot d\vec{l}$

Connection to real space (xy-plane) requires knowledge of: $u(x,y)$, $v(x,y)$

Connection ~~in terms of~~ to original problem in z-space $(xy-)$ in terms of a transformation function: $w = f(z)$.

$u = \text{constant} \xrightarrow{\text{map to}}$ equipotential in real space
$v = \text{constant} \xrightarrow{\text{map to}}$ $\vec{E}$-field line in real space

"just" need correct transformation function

Have already noted $f(z)$ must be analytic for u, v to satisfy Laplace equation.

$\Rightarrow$ derivative independent of path, so: $dw = f'(z)\,dz$

If we represent, $f'(z) = Ae^{i\alpha}$ $\Rightarrow$ $|dw| = |dz| \cdot A$
(in polar form, A real) $dw = Ae^{i\alpha}\,dz$
$\Rightarrow$ (angle of dw) = (angle of dz) + α

$\Rightarrow$ Entire infinitesimal region in neighborhood of point w is similar to infinitesimal region in vicinity of corresponding point z, but magnified by scale factor A, and rotated through angle α.

- Clearly, if two curves C and C' intersect at certain angle in the z-$(xy-)$ plane, (e.g., ϕ and $\vec{E}$-lines $\perp$), transformed curves $\mathfrak{C}$ and $\mathfrak{C}'$ in w-plane will intersect at same angle (e.g., still $\perp$), as both have been rotated by angle α. [same transformation rule $w = f(z)$ applies to both curves!]

$\rightarrow$ Transformations $w = f(z)$ which preserve these angles are said to be conformal
$\Rightarrow$ every analytic function is $\therefore$ a conformal transformation

Problem in Nutshell: Find correct analytic function $w = f(z)$ which transforms sought-after $\underbrace{\Phi - \vec{E} \text{ map}}_{\text{in } z\text{ plane}}$ into simple rectangular grid in (simple problem) w-plane!

Begin With Examples of Inverse (Easier!) Problem: (Knowing $w = f(z)$)

If n is positive real number (not necessarily integer)

$$w = z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

$$\Rightarrow \quad u = r^n \cos n\theta$$
$$v = r^n \sin n\theta$$