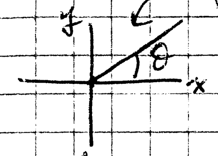
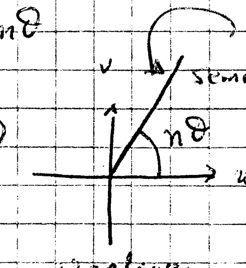
As $w = z^n = (re^{i\theta})^n = r^n e^{in\theta}$



semi-infinite straight line

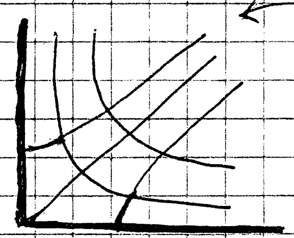
(mapped to →)
[under $w = f(z)$]



semi-infinite straight line

z-plane

w-plane

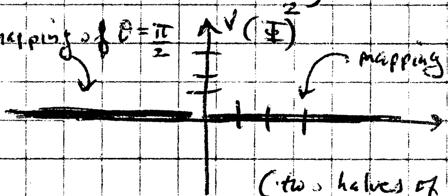


Consider this potential problem: find Φ in $\begin{matrix} x>0 \\ y>0 \end{matrix}$ region.
Boundaries are semi-infinite lines at $\theta = 0, \pi/2$.

Choose V as potential, with $n = 2$.

$$\theta = 0 \Rightarrow \begin{cases} w = r^n e^{i2\cdot 0} = r^n \\ V = r^n \sin(2\cdot 0) = 0 \end{cases} \qquad \begin{matrix}\phi = \\ [\text{angle} = 0 \text{ in } w\text{ plane}]\end{matrix}$$

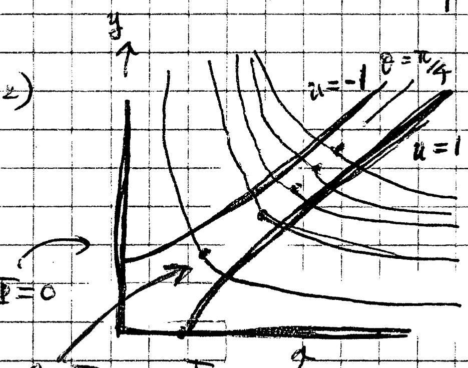
$$\theta = \pi/2 \Rightarrow \begin{cases} w = r^n e^{i\pi} = -r^n \\ V = r^n \sin(2\cdot\frac{\pi}{2}) = 0 \end{cases} \qquad \begin{matrix}\phi = \\ [\text{angle} = \pi \text{ in } w\text{ plane}]\end{matrix}$$



(two halves of $V = 0$ line!)

Then we have: (for $n = 2$)

$$\begin{cases} V = r^2 \sin 2\theta \\ u = r^2 \cos 2\theta \end{cases}$$



⇒ Thus satisfy $V = 0$ (i.e., zero potential) condition

$r = 1,\ \theta = \frac{\pi}{4}$
(closest to origin)

z-plane

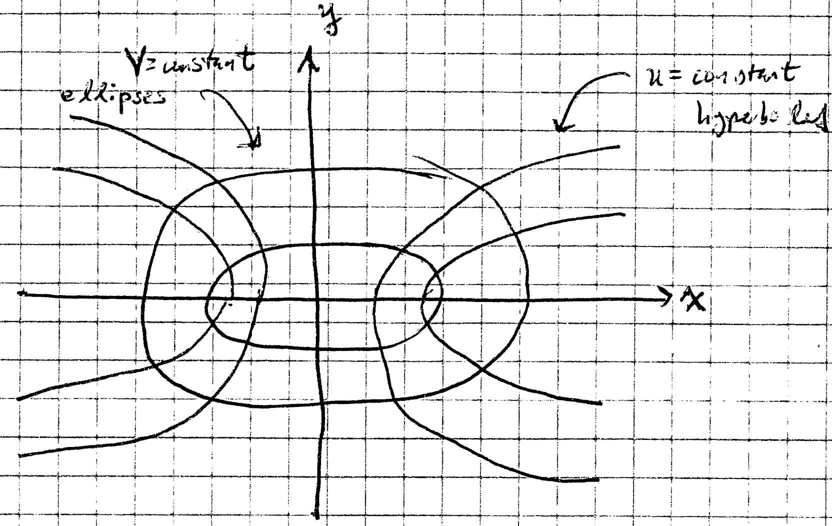
Another example: $w = \cos^{-1} z = \cos^{-1}(x + iy)$ ✱

$$\Rightarrow x + iy = \cos(u + iV) = \cos u \cosh V - i \sin u \sinh V$$

$$\begin{cases} x = \cos u \cosh V \Rightarrow x/\cosh V = \cos u; \quad \cosh V = \dfrac{x}{\cos u} \\ y = -\sin u \sinh V \Rightarrow -y/\sinh V = \sin u; \quad \sinh V = \dfrac{-y}{\sin u} \end{cases}$$

$$\sin^2 u + \cos^2 u = 1 \Rightarrow \frac{x^2}{\cosh^2 V} + \frac{y^2}{\sinh^2 V} = 1$$

$$\cosh^2 V - \sinh^2 V = 1 \Rightarrow \frac{x^2}{\cos^2 u} - \frac{y^2}{\sin^2 u} = 1$$



z-plane

So we can solve many complex problems with this: (examples include)

① if V represents potential,

— field $\vec{E}$ lines between two elliptical cylinders (u)

— field external to a charged elliptic cylinder (u)

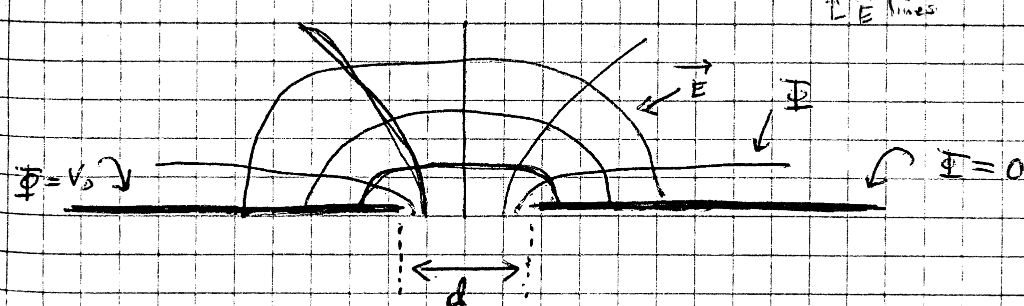
— etc.

② if u represents the potential:

— field between two hyperbolic cylinders (V)

— two charged plates separated by a gap (V)

⊥ $\vec{E}$ lines



So let u represent the potential Φ!

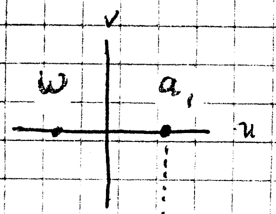
Schwarz Transformation: Real Problem, Not (Easier) Inverse when given $w(z)$

We saw that the function (transformation) $w = z^2$ mapped an angular range of the z-plane $(0, \pi/2)$ into the upper half of the w-plane, bounded by semi-infinite lines at angles of $2\cdot\theta \Rightarrow (0, \pi)$.

More generally, Schwarz transformation maps (interior of) a polygon in the z-plane (need not be closed) to the upper half of the w-plane.

Let a_1 be fixed point on real axis (in w-plane); let $z = g(w)$ be a function whose derivative (inverse) $\frac{dz}{dw}$ is given by:

$$\frac{dz}{dw} = (w-a_1)^{\alpha} \quad \text{for real } \alpha, \ -1 < \alpha < 1$$



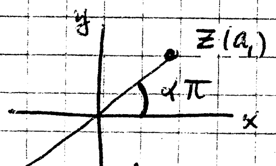
branch cut
$-\frac{\pi}{2} < \arg w < 3\pi/2$

If w on real axis to the "left" of a_1,

- $\frac{dz}{dw}$ exists and is $\neq 0 \Rightarrow$ transformation analytic $\left(\frac{dw}{dz}\right) \neq \infty$!
- $\arg[z'(w)] = \arg[(w-a_1)^{\alpha}] = \alpha\pi$ (constant)

$[\text{if } (w-a_1) \equiv Re^{i\theta}] \Rightarrow (w-a_1)^{\alpha} = R^{\alpha} \cdot e^{i\alpha\theta}$ arg

$\Rightarrow z(w)$ maps the interval $(-\infty, a_1)$ onto a portion of a straight line terminating at $z(a_1)$:

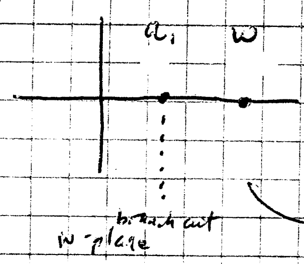


$$\frac{dz}{dw} = (w-a_1)^{\alpha} = R^{\alpha} e^{i\alpha\theta} \propto e^{i\alpha\theta}$$

$$\Rightarrow dz \propto e^{i\alpha\theta} \cdot dw$$

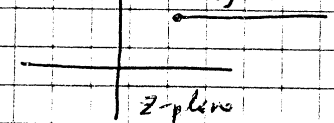
if dw real, $\underbrace{dz \propto (\text{real}) e^{i\alpha\theta} dw}_{\text{(straight line !!)}}$

Similarly, if



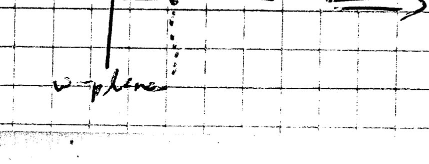
$\arg[z'(w)] = \arg[(w-a_1)^{\alpha}] = \alpha \cdot 0 = 0$ horizontal line

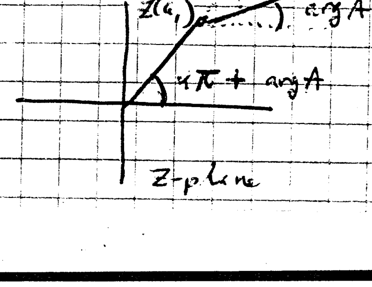
(a_1, ∞) maps to:



Generalizing, if have: $z'(w) = A(w-a_1)^{\alpha}$, A: complex constant, $\neq 0$

$\arg[z'(w)] = \arg[A(w-a_1)^{\alpha}] = \arg A + \alpha \arg(w-a_1)$] to see: consider $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$





Branch Cuts:

Consider $z = re^{i\theta}$

$\Rightarrow \ln z = \ln r + i\theta$

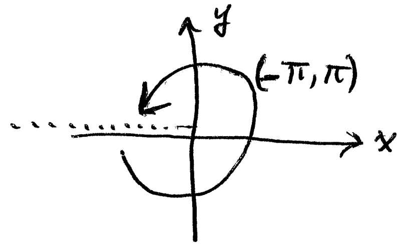
Note: if $\theta \to \theta + 2\pi$, z does not change.

But! $\ln z$ will change $\Rightarrow$ multi-valued function!

$$\ln z = \ln r + i(\theta + 2n\pi) \qquad n = 0, 1, 2, \ldots$$

To avoid ambiguity, simplest choice is $n=0$, limitation to interval of length 2π, such as $(-\pi, \pi)$.

Line in the z-plane that is not crossed, labeled a cut line or branch cut.



value of $\ln z$ with $n=0$ termed "principal value of $\ln z$".