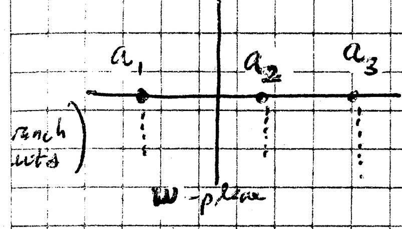
Next generalization:

$$\frac{dz}{dw} = z'(w) = A(w-a_1)^{\alpha_1}(w-a_2)^{\alpha_2}\cdots(w-a_n)^{\alpha_n}$$

$A \neq 0$, complex constant; α_i real, $-1 < \alpha_i < 1$

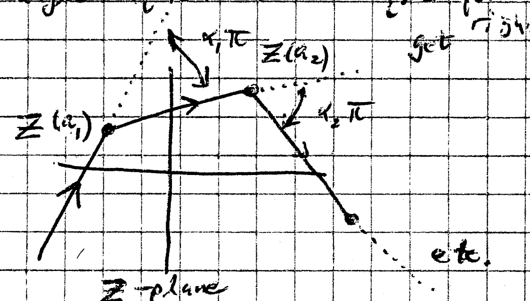
a_i real, $a_1 < a_2 < \dots < a_n$

$$\Rightarrow \arg z'(w) = \arg A + \alpha_1 \arg(w-a_1) + \alpha_2 \arg(w-a_2) + \dots + \alpha_n \arg(w-a_n)$$



Interval on w:	Angle of Mapping in z-plane:
$(-\infty, a_1)$	$\arg A + \alpha_1\pi + \alpha_2\pi + \dots + \alpha_n\pi$
(a_1, a_2)	$\arg A + \alpha_2\pi + \dots + \alpha_n\pi$
(a_2, a_3)	$\arg A + \alpha_3\pi + \dots + \alpha_n\pi$

$\Rightarrow$ As w traverses the real axis from left-to-right, $z(w)$ generates a polygonal path whose tangent at the point $z(a_i)$ makes a "right turn" through an angle $\alpha_i\pi$. (note $\alpha_i > 0$ for get right-hand turns, $\alpha_i < 0$ for left-hand)



Now, since, by construction, $z(w)$ satisfies $z'(w) = A(w-a_1)^{\alpha_1}\dots(w-a_n)^{\alpha_n}$

$$\Rightarrow z(w) = A\int (w-a_1)^{\alpha_1}\dots(w-a_n)^{\alpha_n}\,dw + B \quad \left[\alpha_i = \frac{\theta_i}{\pi}\right]$$

Functions of this form known as Schwarz Transformations,

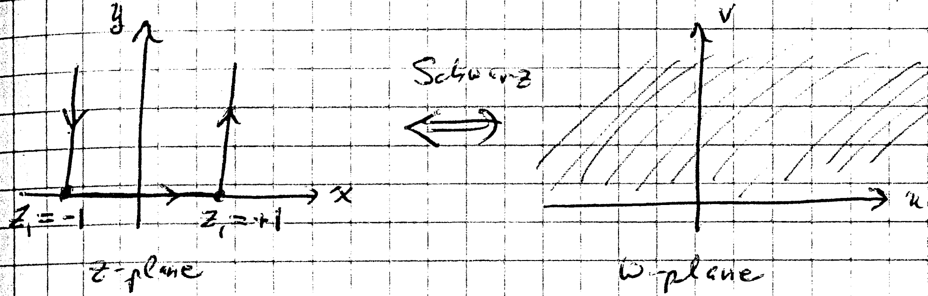
maps: polygon in z-plane $\Leftrightarrow$ real axis in w-plane

Example: Derive a Schwarz transformation which maps the semi-infinite strip, in the z plane

$$|\mathrm{Re}\, z| = |x| < 1$$

$$\mathrm{Im}\, z = y > 0$$

into the upper half-plane in the w-plane.



Choose orientation indicated by arrows:

- Need "left turns" at z_1 & z_2 $\Rightarrow$ $\alpha_1 = \alpha_2 = \frac{-\pi}{2}/\pi = -\frac{1}{2}$.
- choose $a_1 = -1$, $a_2 = +1$ $\Rightarrow$ nothing "magic" about $a_1 = -1$, $a_2 = +1$; same result if use: $a_1 = -2$, $a_2 = +1$

$$\Rightarrow z'(w) = A(w+1)^{-1/2}(w-1)^{-1/2}$$

$$\Rightarrow z(w) = \int dw\, A(w+1)^{-1/2}(w-1)^{-1/2} + B$$

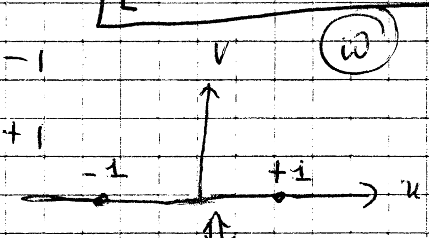
$$= A\int \frac{1}{\sqrt{w^2-1}}\, dw + B = \frac{A}{i}\int \frac{dw}{\sqrt{1-w^2}} + B$$

$$= \frac{A}{i}\sin^{-1} w + B$$

$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x})$$
$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$\cos\left(\tfrac{\pi}{2}\right) = \cos\left(-\tfrac{\pi}{2}\right) = 0$$

Require:

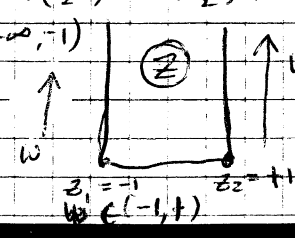
$$\begin{cases} z(a_1) = z(-1) = \frac{A}{i}\sin^{-1}(-1) + B = z_1 = -1 \\ z(a_2) = z(+1) = \frac{A}{i}\sin^{-1}(1) + B = z_2 = +1 \end{cases}$$



$$\left.\begin{aligned} -iA\sin^{-1}(-1) + B &= -1 \\ -iA\sin^{-1}(1) + B &= 1 \end{aligned}\right\} \quad B = 0$$

$$\left[w = \sin\left(\tfrac{\pi}{2}(x+iy)\right) = \sin\left(\tfrac{x\pi}{2}\right)\cosh\left(\tfrac{y\pi}{2}\right) + i\cos\left(\tfrac{x\pi}{2}\right)\sinh\left(\tfrac{y\pi}{2}\right) \right]$$

$$-iA\sin^{-1}(1) = -iA\frac{\pi}{2} = +1 \Rightarrow A = \frac{2i}{\pi}$$



$$\Rightarrow z(w) = \frac{2}{\pi}\sin^{-1} w$$

$$w = \sin\left(\frac{\pi}{2} z\right)$$