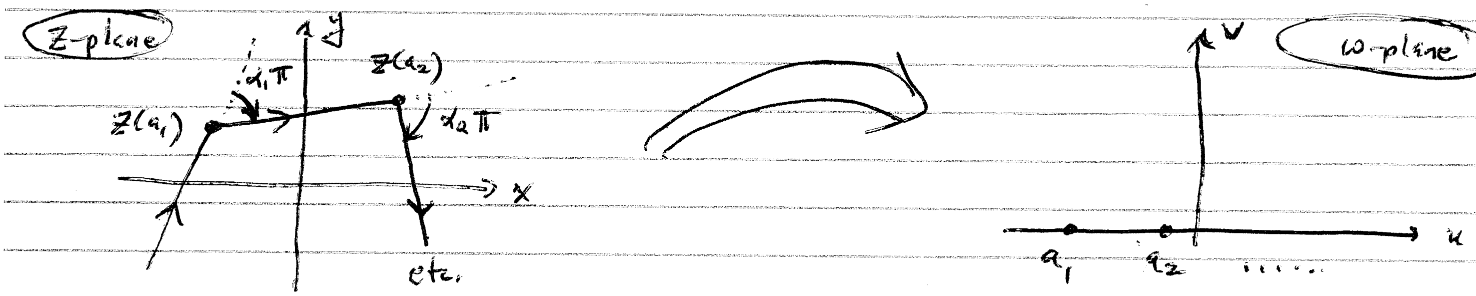
Summary of Schwarz Transformation

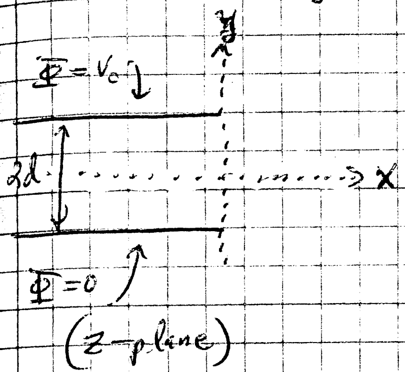
Maps interior of polygon in z-plane into upper half-plane of ω-plane.



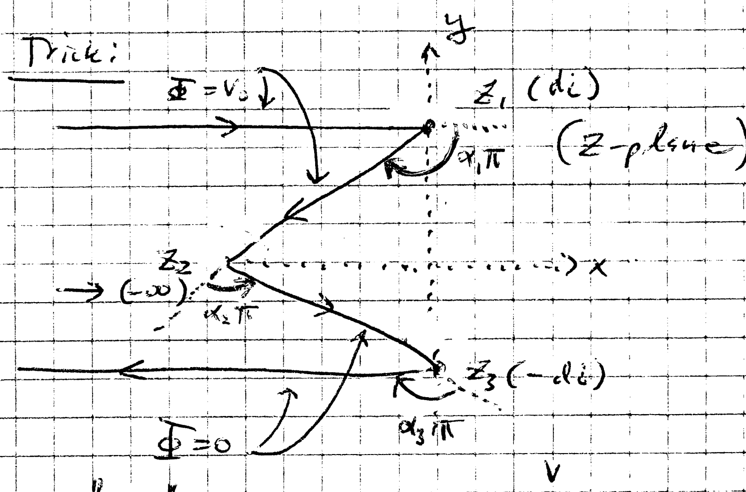
$a_1 < a_2 < a_3 < \ldots\ldots$

$$z(\omega) = A \int (\omega - a_1)^{\alpha_1} (\omega - a_2)^{\alpha_2} \ldots\ldots (\omega - a_n)^{\alpha_n} \, d\omega + B$$

Example: Find the potential Φ for this semi-infinite charged-plate geometry.

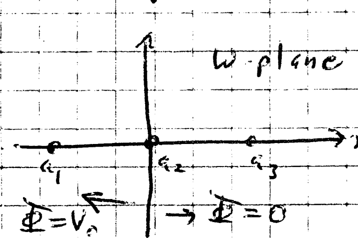


Trick:



In $z_2 \to -\infty$ limit, get these turns:

Right $+\pi$ at z_1 mapping $\iff$ choose: $a_1 = -1$
Left $-\pi$ at z_2 $a_2 = 0$
Right $+\pi$ at z_3 $a_3 = +1$



$$\Rightarrow z'(w) = A(w+1)^{\pi/\pi}(w)^{-\pi/\pi}(w-1)^{\pi/\pi} = A(w+1)^{1}w^{-1}(w-1)^{1}$$

$$\frac{dz}{dw} = A\frac{1}{w}\cdot(w^2-1) = A\left(w - \frac{1}{w}\right)$$

$$\Rightarrow z(w) = A\left(\tfrac{1}{2}w^2 - \ln w\right) + B$$

$$\left[\begin{aligned} &\ln z = \ln(re^{i\theta}) \\ &= \ln r + i\theta \end{aligned}\right]$$

Enforcing:
$$\begin{cases} z(a_1) = z(-1) = A\left(\tfrac{1}{2} - \ln(-1)\right) + B = +di \\ z(a_3) = z(1) = A\left(\tfrac{1}{2} - \ln(1)\right) + B = -di \end{cases}$$

$$\begin{cases} A\left(\tfrac{1}{2} - i\pi\right) + B = +di \\ \tfrac{1}{2}A + B = -di \end{cases} \Rightarrow -\cancel{i}A\pi = 2\cancel{i}d$$

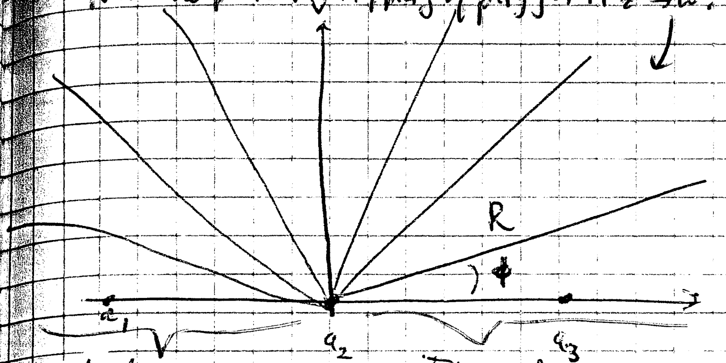
for branch cut $\Rightarrow (-\pi, \pi)$

$$\Rightarrow A = -\frac{2d}{\pi}$$

$$B = -di + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}d}{\pi} = -di + \frac{d}{\pi}$$

Mapping is:
$$z(w) = -\frac{2d}{\pi}\left(\tfrac{1}{2}w^2 - \ln w\right) + \frac{d}{\pi} - di$$
$$= \underline{\underline{\frac{d}{\pi}\left(1 - w^2\right) + \frac{2d}{\pi}\ln w - di}}$$

Now in ω-plane: mapping of polygon in $z \to \omega$:



$\Phi = V_0$ for $u < 0, v = 0$ $\quad$ $\Phi = 0$ for $u > 0, v = 0$

(ω-plane)

BEFORE, said "if" u or v can be chosen to represent Φ.....

$$\Phi = V_0 \cdot \frac{\phi}{\pi},$$

$\Rightarrow$ ϕ constant traces out equipotentials Φ

$\Downarrow$

[Note, here cannot simply identify either u or v as a constant Φ $\Rightarrow$ more complicated!]

Equipotentials: $\quad$ constant $= V_0 \cdot \frac{\phi}{\pi}$,

parametrize: $\quad$ $\Rightarrow$ ϕ must be constant, $\omega = Re^{i\phi}$, ϕ constant, $0 < R < \infty$

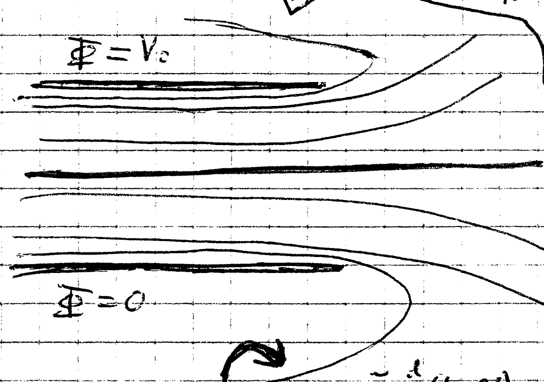
[difficult to directly invert as Φ is not direct function of ω, so no easy form for $z(\Phi(\omega))$.]

$\Downarrow$

insert into $z(\omega) = \frac{d}{\pi}(1-\omega^2) + \frac{2d}{\pi}\ln\omega - di$

$= \frac{d}{\pi}(1-\omega^2) + \frac{2d}{\pi}\ln R + \frac{2d}{\pi} i\phi - di$

$\Downarrow$



$\phi = \pi/2$:

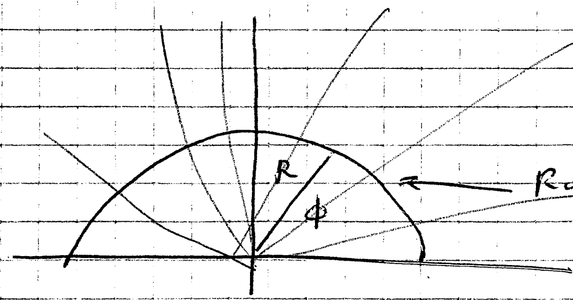
$\omega = Re^{i\pi/2}$

$= Ri$

$\Rightarrow z = (-\infty, \infty) + 0i$

$\phi = \varepsilon$: $\quad \omega = Re^{i\varepsilon}$, $\varepsilon \ll 1$; $\quad z \cong \frac{d}{\pi}(1-R^2) - \frac{2dR^2\varepsilon}{\pi} i + \frac{2d}{\pi}\ln R + i\left(\frac{2d\varepsilon}{\pi} - d\right)$

$\Rightarrow$ R constant in $\omega = Re^{i\phi}$ traces out E-line



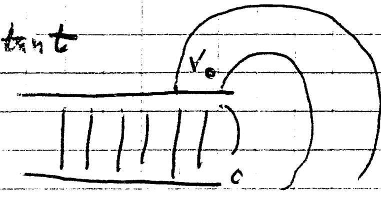
R constant, $0 < \phi < \pi$

$\Downarrow$

$\omega = Re^{i\phi}$, R constant

$\Downarrow$

$z(\omega)$ $\Rightarrow$



$\Rightarrow$ conformal, $\Phi \perp E$ in ω- and z-planes!