Laplace Equation in Spherical Coordinates

In (r,θ,ϕ) coordinates, $\vec{\nabla}^2 \Phi(\vec{x}) \overset{0}{=}$ is:

$\frac{\partial\phi}{\partial r}$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Assuming separation of variables: $\Phi = \frac{U(r)}{r}\,P(\theta)\,Q(\phi)$

Substituting:

$$PQ\frac{1}{r}\frac{d^2U}{dr^2} + \frac{1}{r^2\sin\theta}\frac{UQ}{r}\frac{d}{d\theta}\left(\sin\theta\cdot\frac{dP}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{U}{r}\cdot P\cdot\frac{d^2Q}{d\phi^2} = 0$$

Multiply by: $\frac{1}{UPQ}\cdot r^3\sin^2\theta$

$$r^2\frac{1}{U}\sin^2\theta\,\frac{d^2U}{dr^2} + \frac{1}{P}\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0$$

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)\right] + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\phi\text{-dependence only in this term now!}} = 0$$

Thus, $\frac{1}{Q}\frac{d^2Q}{d\phi^2}$ must be a constant, because for fixed (r,θ), $[\ldots]$ term will be constant, so if $\frac{1}{Q}\frac{d^2Q}{d\phi^2}$ varies with ϕ, won't solve Laplace equation!

$$\Rightarrow \quad \underline{\underline{\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2}} \quad \Rightarrow \quad Q = e^{\pm im\phi}, \quad \text{allowed } m \text{ values?}$$

Jackson: "For Q to be single valued, m must be an integer if the full azimuthal range is allowed."

Why? e.g.,

		$\phi = 0$	$\phi = 2\pi$
$0 \le \phi \le 2\pi$, if $m = 1$:	$\cos\phi + i\sin\phi$	1	1
what, e.g., if $m = 1.5$:	$\cos(1.5\phi) + i\sin(1.5\phi)$	1	-1 ✗

multi-valued!

$\Rightarrow$ m must be an integer: $-\infty, \ldots, +\infty$

So now we have for the Laplace Equation:

$$\left(r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right)\right] - m^2 = 0\right)\cdot\frac{1}{\sin^2\theta}$$

$$\Rightarrow \quad \underbrace{r^2\frac{1}{U}\frac{d^2U}{dr^2}}_{r\text{ only}} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\theta\text{-only}} = 0$$

Note that:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{r}\frac{\partial}{\partial r}\left(\psi + r\frac{\partial\psi}{\partial r}\right)$$

$$= \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{\cancel{r}}\cancel{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r^2}2r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}r^2\frac{\partial^2\psi}{\partial r^2}$$

$$= \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2}$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \checkmark$$

Now, θ-dependence all contained in:

$$\frac{1}{P(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\equiv -\ell(\ell+1) \qquad \begin{matrix}\ell(\ell+1)\\ \text{(real constant)}\end{matrix}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\cdot P(\theta)=-\ell(\ell+1)\cdot P(\theta)$$

$$\underline{\underline{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\cdot P(\theta)=0}}$$

With these definitions, it then remains (only) that:

$$\frac{r^2}{U}\frac{d^2U}{dr^2}-\ell(\ell+1)=0$$

$$\underline{\underline{\frac{d^2U}{dr^2}-\frac{\ell(\ell+1)U(r)}{r^2}=0}}$$

Have now separated out (r,θ,ϕ) into 3 different differential equations!

Can guess at solutions: (from form of)

$U(r)\propto r^{\ell+1}$ works: $(\ell+1)\ell\, r^{\ell-1}-\frac{\ell(\ell+1)r^{\ell+1}}{r^2}=\ell(\ell+1)r^{\ell-1}-\ell(\ell+1)r^{\ell-1}=0$ ✓

$\propto r^{-\ell}$ works: $(-\ell)(-\ell-1)r^{-\ell-2}-\frac{\ell(\ell+1)r^{-\ell}}{r^2}=\ell(\ell+1)r^{-\ell-2}-\ell(\ell+1)r^{-\ell-2}=0$ ✓

Superposition: $U(r)=Ar^{\ell+1}+Br^{-\ell}$, [$\ell$ not yet determined]

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Now, turning to the θ differential equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\cdot P(\theta)=0 \qquad \left[\frac{d}{d\theta}=\frac{dx}{d\theta}\cdot\frac{d}{dx}\right]$$

Customarily: $x=\cos\theta \Rightarrow dx=-\sin\theta\, d\theta \Rightarrow \frac{d}{d\theta}=-\sin\theta\frac{d}{dx}$

$$\Rightarrow -\frac{d}{dx}\left(-(1-x^2)\frac{dP}{dx}\right)+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]\cdot P(x)=0$$

$$\underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0}$$

"Generalized Legendre Equation" → solutions: Associated Legendre functions

"Ordinary Legendre Equation" has $m=0$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\ell(\ell+1)P=0$$

~~Just state facts (no proofs):~~

Solutions are the Legendre polynomials of order l; $P_l(x)$. $l =$ integer (0, positive)

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$
$$\vdots$$

Note: $P_l(x)$ is even (odd) about $x = 0$ if l is even (odd)

Normalised so: $P_l(1) = 1 \;\forall\, l$

In general, compact representation of the Legendre polynomials via Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2 - 1)^l$$

Form a complete set of orthogonal functions on the interval $-1 \le x \le 1$.

Orthogonality condition:

$$\int_{-1}^{+1} P_{l'}(x)\, P_l(x)\, dx = \frac{2}{2l+1}\, \delta_{l'l}$$

$\Rightarrow$ Can define orthonormal functions:

$$U_l(x) = \sqrt{\frac{2l+1}{2}}\, P_l(x) \Rightarrow \int_{-1}^{+1} U_{l'}(x)\, U_l(x)\, dx = \sqrt{\frac{2l'+1}{2}}\sqrt{\frac{2l+1}{2}} \cdot \int_{-1}^{+1} P_{l'}(x)\, P_l(x)\, dx = \delta_{l'l}$$

Thus, can expand any function $f(x)$ on the interval $-1 \le x \le 1$ as:

recall in general:

$$f(x) = \sum_{l=0}^{\infty} a_l\, U_l(x) \quad ; \quad a_l = \int dx\, f(x)\, U_l(x)$$

$$\Rightarrow \quad a_l = \int_{-1}^{+1} \sqrt{\frac{2l+1}{2}}\, P_l(x) \cdot f(x)\, dx$$

$$\Rightarrow f(x) = \sum_{l=0}^{\infty} \left[\int_{-1}^{+1} \sqrt{\frac{2l+1}{2}}\, P_l(x)\, f(x)\, dx \right] \cdot \underbrace{\sqrt{\frac{2l+1}{2}}\, P_l(x)}_{U_l(x)}$$

$$= \sum_{l=0}^{\infty} \left(\frac{2l+1}{2}\right) \left[\int_{-1}^{+1} P_l(x)\, f(x)\, dx \right] P_l(x)$$

$$\equiv \sum_{l=0}^{\infty} C_l\, P_l(x), \quad C_l = \left(\frac{2l+1}{2}\right) \int_{-1}^{+1} P_l(x)\, f(x)\, dx$$

Recurrence Relations: useful for evaluating integrals, generating higher-order $P_l(x)$ from lower-order, etc...

$$(l+1) P_{l+1} - (2l+1)\, x\, P_l + l P_{l-1} = 0$$

$$\frac{dP_{l+1}}{dx} - x \frac{dP_l}{dx} - (l+1) P_l = 0$$

$$(x^2 - 1) \frac{dP_l}{dx} - l x P_l + l P_{l-1} = 0$$

Example: $$I = \int_{-1}^{+1} x P_\ell(x) P_{\ell'}(x)\, dx$$

From recurrence relation:

$$(\ell+1)P_{\ell+1} - (2\ell+1) x P_\ell + \ell P_{\ell-1} = 0$$

$$x P_\ell = \frac{1}{(2\ell+1)}\left[(\ell+1) P_{\ell+1} + \ell P_{\ell-1}\right]$$

$$\Rightarrow I = \int_{-1}^{+1} \frac{1}{(2\ell+1)}\left[(\ell+1) P_{\ell+1} P_{\ell'} + \ell P_{\ell-1} P_{\ell'}\right] dx$$

$$= \frac{1}{(2\ell+1)}\left[(\ell+1)\, \delta_{\ell+1,\ell'} \frac{2}{2\ell'+1} + \ell \cdot \frac{2}{2\ell'+1} \cdot \delta_{\ell-1,\ell'}\right]$$

$$\left[\int_{-1}^{+1} dx\, P_m(x) P_n(x) = \delta_{mn} \cdot \frac{2}{2m+1}\right]$$

$$= \frac{1}{(2\ell+1)} \cdot \begin{cases} \dfrac{2(\ell+1)}{(2\ell+3)} & \text{if } \ell' = \ell+1 \\ \dfrac{2(\ell)}{(2\ell-1)} & \text{if } \ell' = \ell-1 \end{cases}$$

Boundary-Value Problems with Azimuthal Symmetry:

"Apply" Legendre polynomials to solution of Laplace equation under azimuthal symmetry.

By azimuthal symmetry, we mean no ϕ-dependence. As we had

$$\Phi(r,\theta,\phi) = \frac{U(r)}{r} \cdot P(\theta) \cdot Q(\phi), \text{ with } Q(\phi) = e^{\pm i m \phi}, \text{ need } m=0 \text{ for no } \phi\text{-dependence}$$

$\Downarrow$

P solutions are the Legendre Polynomials

$$\Rightarrow \Phi(r,\theta) = \sum_{\ell=0}^{\infty} \overbrace{\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]}^{= u(r)/r} P_\ell(\cos\theta)$$

A_ℓ, B_ℓ: determined from boundary conditions

Example. potential $V(\theta)$ specified on the surface of a sphere of radius R; find Φ inside the sphere, assuming no charges inside sphere.

- First, clearly need $B_\ell = 0$, else $\Phi \to \infty$ at $r=0$ (since no q at $r=0$)

$$\Rightarrow \Phi(R,\theta) = V(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

This is just the expansion of $f(x) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(x)$ we had before, but with

$$C_\ell \to A_\ell R^\ell$$

$$\left| \; C_\ell = \left(\frac{2\ell+1}{2}\right) \cdot \int_{-1}^{+1} P_\ell(x) f(x)\, dx \right.$$

So, now, $$A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi P_\ell(\cos\theta) V(\theta) \sin\theta\, d\theta$$