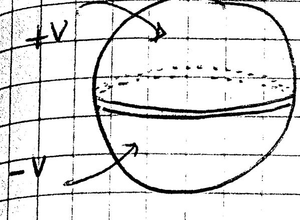
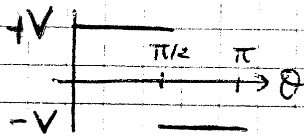
Recall old problem:

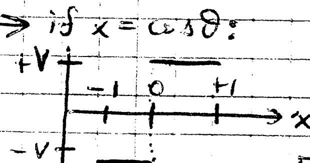


Past, solved via Green Function technique. ↙ analytic expression on z-axis

$$V(\theta) = \begin{cases} +V & 0 \le \theta \le \pi/2 \\ -V & \pi/2 \le \theta \le \pi \end{cases}$$



→ if $x = \cos\theta$:



$f(-x) = -f(x)$

so $V(x)$ is odd on the interval $-1 \le x \le 1$

⇒ only odd P_ℓ have non-zero coefficients

Find $\Phi(\vec{x})$ everywhere in space:

Use B.C. to find A_ℓ

In: $V(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$

$$\underbrace{A_\ell}_{\text{odd only}} = \frac{2\ell+1}{2R^\ell}\int_0^\pi V(\theta)\cdot P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{2\ell+1}{2R^\ell}\cdot\left[V\int_0^{\pi/2} P_\ell(\cos\theta)\sin\theta\, d\theta - V\int_{\pi/2}^{\pi} P_\ell(\cos\theta)\sin\theta\, d\theta\right]$$

$$= \frac{(2\ell+1)V}{2R^\ell}\left[\int_0^1 P_\ell(x)\,dx - \int_{-1}^0 P_\ell(x)\,dx\right]$$

As ℓ is odd ⇒ P_ℓ is odd in x ⇒ $-\int_{-1}^0 P_\ell(x)\,dx = +\int_0^1 P_\ell(x)\,dx$

[easy to see; e.g., $P_1(x) = x$]

$$\Rightarrow \underset{\text{odd }\ell}{A_\ell} = \frac{(2\ell+1)V\cdot\cancel{2}}{\cancel{2}R^\ell}\cdot\underbrace{\int_0^1 P_\ell(x)\,dx}_{\text{evaluate this integral } \forall \ell} = \frac{V(2\ell+1)}{R^\ell}\int_0^1 P_\ell(x)\,dx$$

$$\left[P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell \quad (\text{by Rodrigues Formula})\right]$$

Result is quoted in Jackson:

$$\int_0^1 P_\ell(x)\,dx = \frac{1}{(2\ell+1)}\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}\frac{(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$\Rightarrow A_\ell = \frac{V}{R^\ell}\left(-\frac{1}{2}\right)^{\frac{\ell-1}{2}}\frac{(2\ell+1)(\ell-2)!!}{2\left(\frac{\ell+1}{2}\right)!}$$

$$\ell = 1: \quad A_\ell = \frac{V}{R}\left(-\frac{1}{2}\right)^0\frac{(3)(-1)!!}{2(1)!} = \frac{3}{2}\frac{V}{R}$$

$$\ell = 3: \quad A_\ell = \frac{V}{R^3}\left(-\frac{1}{2}\right)^1\frac{(7)(1)!!}{2(2)!} = -\frac{7}{8}\frac{V}{R^3}$$

Need $\int_0^1 P_\ell(x)\,dx$ for odd ℓ: (could evaluate each "by hand", but let's develop a recurrence relation)

Using recurrence: $(m+1)P_m = \frac{dP_{m+1}}{dx} - x\frac{dP_m}{dx}$

$$(m+1)\int_0^1 P_m(x)\,dx = \int_0^1 \frac{dP_{m+1}}{dx}\,dx - \underbrace{\int_0^1 x\frac{dP_m}{dx}\,dx}_{\substack{u=x,\ du=dx \\ dv=\frac{dP_m}{dx},\ v=P_m}}$$

$$= P_{m+1}(x)\Big|_0^1 - xP_m(x)\Big|_0^1 + \int_0^1 P_m(x)\,dx$$

$$m\int_0^1 P_m(x)\,dx = P_{m+1}(x)\Big|_0^1 - xP_m(x)\Big|_0^1$$

$$\left[\forall \ell,\ P_\ell(1) = 1\right]$$

$$= -P_{m+1}(0)$$

Using recurrence: $(m+1)P_{m+1} - (2m+1)xP_m + mP_{m-1} = 0$

$$P_{m+1} = \frac{(2m+1)xP_m - mP_{m-1}}{(m+1)}$$

$$x=0:\quad P_{m+1}(0) = \frac{-m\,P_{m-1}(0)}{m+1}$$

$$\Rightarrow \int_0^1 P_m(x)\,dx = +\frac{P_{m-1}(0)}{m+1}$$

So, for ℓ odd, start with:

$\underline{\underline{\ell=1}}$: $\int_0^1 P_1(x)\,dx = +\frac{P_0(0)}{2} = +\frac{1}{2}$

$$\Rightarrow A_\ell = \frac{3\cdot V}{R^1}\cdot\left(+\frac{1}{2}\right) = \frac{3}{2}\frac{V}{R} \quad ✓$$

Next, $\underline{\underline{\ell=3}}$:

$$\int_0^1 P_3(x)\,dx = +\frac{P_2(0)}{4} = \frac{1}{4}\cdot\left[\frac{-1\cdot P_0(0)}{2}\right] = -\frac{1}{8}$$

$$\Rightarrow A_3 = \frac{(7)}{R^3}V\left(-\frac{1}{8}\right) = -\frac{7}{8}\frac{V}{R^3} \quad ✓$$

Next, $\underline{\underline{\ell=5}}$:

$$\int_0^1 P_5(x)\,dx = +\frac{P_4(0)}{6} = \frac{1}{6}\cdot\left[\frac{-3\,P_2(0)}{4}\right] = \frac{-3}{24}\cdot\left[-\frac{1}{2}\,P_0(0)\right]$$

$$= \frac{-3}{24}\cdot\left(-\frac{1}{2}\right)(1) = +\frac{3}{48} \Rightarrow A_5 = \frac{(11)V}{R^5}\cdot\left(+\frac{3}{48}\right) = +\frac{33}{48}\frac{V}{R^5} = +\frac{11}{16}\frac{V}{R^5}$$

Now that we have the A_ℓ:

$$\Phi(r,\theta)\Big|_{in} = \sum_{\ell=1,\, odd}^{\infty} A_\ell\, r^\ell\, P_\ell(\cos\theta)$$

$$= \frac{3}{2}\frac{V}{R}\cdot r\, P_1(\cos\theta) - \frac{7}{8}\frac{V}{R^3} r^3 P_3(\cos\theta) + \frac{11}{16}\frac{V}{R^5} r^5 P_5(\cos\theta) + \ldots$$

This is the potential inside the sphere:

Now, what about outside the sphere?

Now, demand in: (general form for solution):

$$\Phi(r,\theta) = \sum_{\ell=0,\, odd}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

clearly Need $A_\ell = 0$, else $\Phi \to \infty$ at $r \to \infty$, $\Phi(r,\theta) = \sum_{\substack{\ell=0\\ odd}}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

$$\Phi(R,\theta) = V(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\left[f(x) = \sum_{\ell=0}^{\infty} c_\ell P_\ell(x), \quad c_\ell = \left(\frac{2\ell+1}{2}\right)\int_{-1}^{+1} P_\ell\, f(x)\, dx \right]$$

just same as before, but now with: $c_\ell \to \frac{B_\ell}{R^{\ell+1}}$

$$B_\ell = \left(\frac{2\ell+1}{\cancel{2}}\right) R^{\ell+1} \cancel{2} V \int_0^1 P_\ell(x)\, dx = (2\ell+1) R^{\ell+1} V \int_0^1 P_\ell(x)\, dx$$

$$= (R^{\ell+1}) R^\ell \cdot A_\ell$$

ℓ-odd as before for B.C.:

$$\Rightarrow B_1 = R^2 \cdot R^1 \frac{3}{2}\frac{V}{R} = \frac{3}{2} V R^2$$

$$B_3 = R^4 \cdot R^3 \cdot \left(-\frac{7}{8}\frac{V}{R^3}\right) = -\frac{7}{8} V R^4$$

$$B_5 = R^6 R^5 \cdot \left(\frac{11}{16}\frac{V}{R^5}\right) = \frac{11}{16} V R^6$$

$$\Phi(r,\theta)\Big|_{outside} = \frac{3}{2} V R^2 \cdot \frac{1}{r^2} P_1(\cos\theta) - \frac{7}{8} V R^4 \frac{1}{r^4} P_3(\cos\theta) + \frac{11}{16} V R^6 \cdot \frac{1}{r^6} P_5(\cos\theta) + \ldots$$

Note: At $r=R$:

$$\Phi_{inside}(r=R,\theta) = \Phi_{outside}(r=R,\theta) \quad \checkmark$$

Note: if $0 < r < r_{max}$ where $r_{max} < \infty$, both A_ℓ & B_ℓ can be non-zero!