In >1 dimension:

$$\delta(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\int_V \delta(\vec{x}-\vec{x}')\,d^3x = \begin{cases} 1 & \text{if } V \text{ contains } \vec{x}=\vec{x}' \\ 0 & \text{if not} \end{cases}$$

<u>Note:</u> delta function has dimensions of an inverse volume in whatever number of dimensions the space has

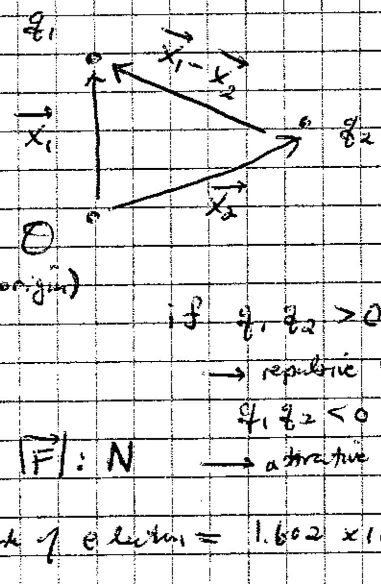
<u>Next Time:</u> start electrostatics, Chapter 1

$\boxed{\text{Lecture \#2}}$ Electrostatics: <u>time-independent</u> distributions of charge and fields

Recall from undergraduate E&M:

Coulomb's Law: The force $\vec{F}$ on a point charge q_1 located at $\vec{x}_1$, due to another point charge q_2 located at $\vec{x}_2$ is

$$\vec{F} = k\,q_1 q_2 \frac{\vec{x}_1-\vec{x}_2}{|\vec{x}_1-\vec{x}_2|^3}$$ (inverse square law)



In SI units, $k = \dfrac{1}{4\pi\varepsilon_0}$

$$\varepsilon_0 = 8.854\,187\,817\ldots \times 10^{-12}\ \frac{F}{m}$$ (F: farad)

"permittivity of free space"

if $q_1 q_2 > 0$ (same sign) $\to$ repulsive!

$q_1 q_2 < 0$ (opposite sign) $\to$ attractive!

q_1, q_2: in Coulombs (C), $|\vec{x}_1|$: m, $|\vec{F}|$: N

[Note: 1 C is huge charge, charge magnitude of electron $= 1.602\times10^{-19}$ C]

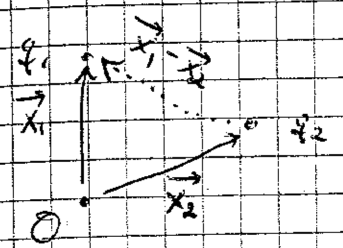
Now, the electric field $\vec{E}$ can be defined as the force per unit charge acting at a given point.

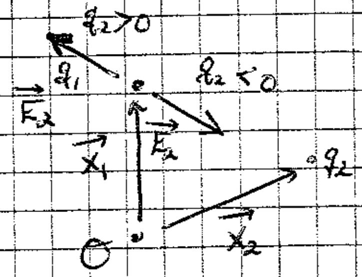
$$\vec{F} = q\vec{E} \Longrightarrow$$

by comparing to Coulomb's Law

Electric field at arbitrary point $\vec{x}$ due to the point charge q_2 at $\vec{x}_2$ is:

$$\vec{E}_2(\vec{x}) = k\,q_2 \frac{\vec{x}-\vec{x}_2}{|\vec{x}-\vec{x}_2|^3}$$





At $\vec{x} = \vec{x}_1$, $\vec{E}_2(\vec{x}_1) = k q_2 \dfrac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$

due to q_2

Units of electric field: SI: $\frac{\text{Volts}}{m}$

To set the scale:
electron produces field of magnitude $1.44 \times 10^{-9} \frac{V}{m}$ at distance of $1\ m$

← see supplement page

So far, considered only one point charge. Generalizing to a system of N point charges $q_1, q_2, \ldots, q_N$, located at $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N$, by the principle of superposition:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

If, instead, of point charges, we have a charge density $\rho(\vec{x}')$ at some $\vec{x}'$, [3-D: SI: $\frac{C}{m^3}$]

discrete sum → integral, $dq' = \rho(\vec{x}')\, d^3x'$

$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3x'}$$

$\vec{x}'$: dummy variable of integration

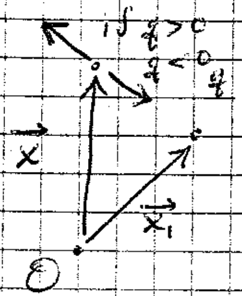
point in question

$[d^3x' = dx'dy'dz'$ or other forms for cylindrical/spherical$]$

This is the most general form. If we have a system of discrete point charges, we can write the charge density at some point $\vec{x}'$ as:

$$\rho(\vec{x}') = \sum_{i=1}^{N} q_i\, \delta(\vec{x}' - \vec{x}_i)$$

For a single point charge q at position $\vec{x}_1$:



At arbitrary $\vec{x}$, $\vec{E}$ is directed along $\pm(\vec{x}-\vec{x}_1)$, depending on the sign of q $(\pm)$.

$\Rightarrow$ the $\vec{E}$ field from a point charge points radially outward/inward.

if: $\vec{r} = \vec{x} - \vec{x}_1$

or $$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$\vec{r}$: vector from charge to point in question

Inserting into integral expression for $\vec{E}$ we find:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sum_{i=1}^{N} q_i\, \delta(\vec{x}' - \vec{x}_i)(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, d^3x'$$

— $\vec{x}'$: just dummy variable of integration

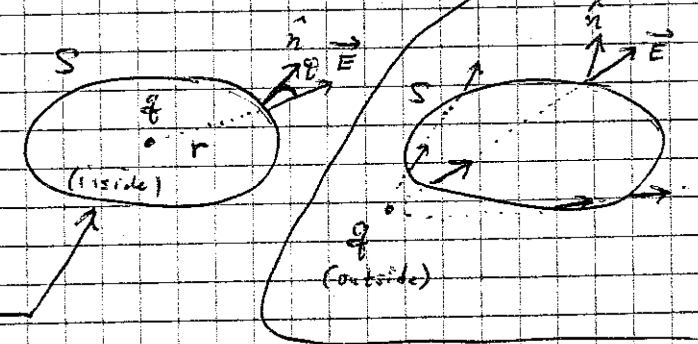
$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\left[q_i \int \frac{\delta(\vec{x}'-\vec{x}_i)(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, d^3x' \right]$$

[δ-function forces $\vec{x}' = \vec{x}_i$]

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3} \quad \checkmark$$

As no doubt recall from undergrad E&M, calculating $\vec{E}$ from general integral expression can be very difficult. Instead, Gauss's law is sometimes more useful!

Suppose we have some (3-D) surface S,



Consider charge inside the surface:

r : distance from q to surface

$\hat{n}$: outwardly directed unit normal ($\perp$ to S)

θ : angle between $\vec{E}$ and $\hat{n}$

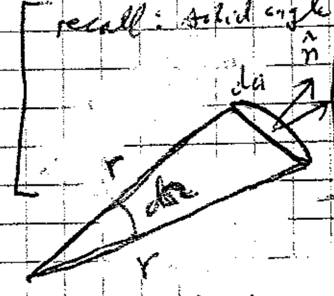
We can immediately write down for an infinitesimal surface area element da:

$$\vec{E}\cdot\hat{n}\, da = \left(\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}\right)\cdot\hat{n}\, da$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\underbrace{\hat{r}\cdot\hat{n}\, da}_{= r^2 d\Omega}$$

$$= \frac{q}{4\pi\varepsilon_0}\, d\Omega$$

[recall: solid angle

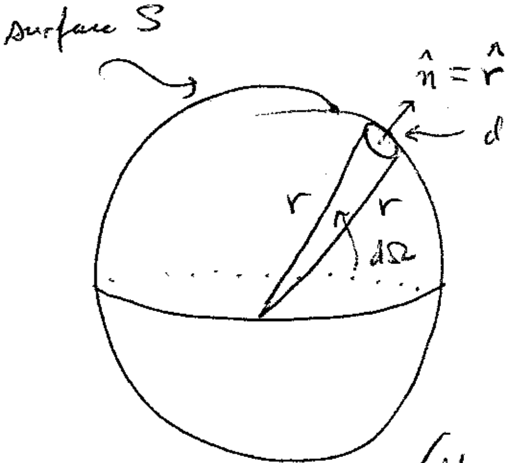


]

$$d\Omega \equiv \frac{\hat{r}\cdot\hat{n}\, da}{r^2}$$

(sphere) if $\hat{r}\cdot\hat{n} = 1 \Rightarrow d\Omega = \frac{da}{r^2}$

Solid Angle Subtended by Infinitesimal Surface Area:

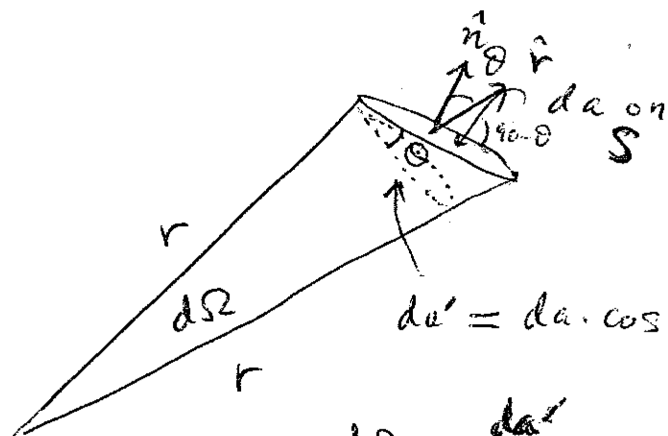


Sphere:

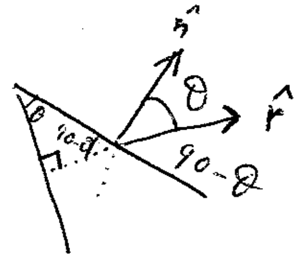
$$d\Omega = \frac{da}{r^2}$$

($\nparallel$ not parallel to)

What if $\hat{n} \nparallel \hat{r}$?



$da' = da \cdot \cos\theta$



$$d\Omega = \frac{da'}{r^2}$$

$$= \frac{da\cos\theta}{r^2}$$

$$= \frac{da \cdot \hat{n}\cdot\hat{r}}{r^2} \checkmark$$

Integrate over the closed surface:

total solid angle subtended by closed surface $= 4\pi$!!

$$\Rightarrow \underbrace{\oint_S \vec{E}\cdot\hat{n}\,da}_{\text{net flux through closed surface}} = \frac{q}{4\pi\epsilon_0}\underbrace{\int_S d\Omega}_{=4\pi} = \frac{q}{\epsilon_0}$$

(if q inside the closed surface)

If q is not inside the closed surface:

$$\oint_S \vec{E}\cdot\hat{n}\,da = 0$$

[every field line that enters the surface must later exit the surface $\Rightarrow$ net flux $= 0$!]

Generalizing:

$$\oint_S \hat{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\sum_i q_i$$

~~for discrete set of point charges~~

(Σ over only those charges inside)

For continuous charge density $\rho(\vec{x})$:

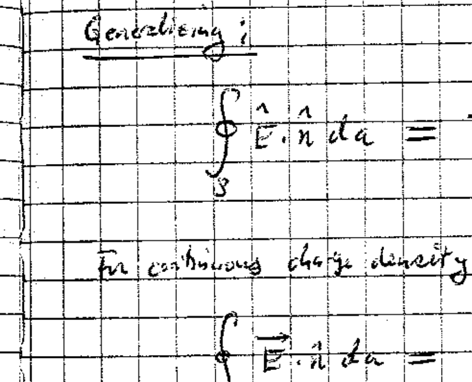
$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\,d^3x$$

[Integral Form of Gauss's Law]

(volume enclosed by S)

Example: sphere of radius a has a spherically symmetric charge density that varies as

Jackson 1.4: r^{-2}, with total charge Q. Find $\vec{E}$ inside and outside the sphere.



Use Gauss's Law: $\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int \rho(\vec{x})\,d^3x$

① Find ρ:

$$Q = \int_V \frac{N}{r^2}\,d^3x$$

use spherical coordinates: $d^3x = r^2\sin\theta\,d\theta\,d\phi\,dr$

$$Q = \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^a N\,dr$$

$$= (2\pi)(2)(Na) = 4\pi N a \Rightarrow N = \frac{Q}{4\pi a}$$

radius $= a$

total charge Q

$$\rho(r) \propto \frac{1}{r^2} = \frac{N}{r^2}$$

N: normalization factor

$$\rho(r) = \frac{Q}{4\pi a r^2}$$

② Find $\vec{E}$: by symmetry, $\vec{E} = E(r)\hat{r}$, $\hat{n} = \hat{r}$ for sphere

<u>$r<a$:</u> define the "pillbox"

$$\oint_S \vec{E}\cdot\hat{n}\,da = \oint_S E(r)\,da = \frac{1}{\epsilon_0}\int_V \rho(r)\,d^3x, \qquad d^3x = r'^2\sin\theta\,dr'\,d\theta\,d\phi$$

$$da = r^2\sin\theta\,d\theta\,d\phi$$

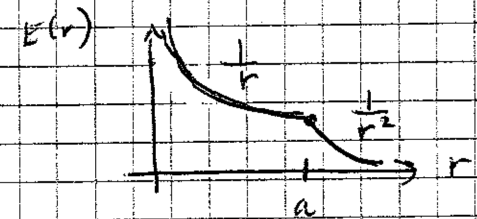
$$\int_0^{2\pi} d\phi\int_0^{\pi} r^2\sin\theta\cdot E(r)\cdot d\theta = \frac{1}{\epsilon_0}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\int_0^{r}\frac{Q}{4\pi a r'^2}r'^2\,dr'$$

$$r^2\cdot E(r)\cdot\cancel{2\pi}\cdot 2 = \cancel{(2\pi)}(2)\cdot\frac{Q}{4\pi a\epsilon_0}r$$

$$E(r) = \frac{Q}{4\pi a r\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 a r}\hat{r}}$$ ✓

<u>$r>a$:</u>

$$E(r)\cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}}$$ radius a is not relevant



<u>Differential Form of Gauss's Law:</u>

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int_V \rho(\vec{x})\,d^3x$$

<u>Recall Gauss's Theorem: Divergence</u> $\quad \oint_S \vec{V}\cdot d\vec{a} = \int_V(\vec{\nabla}\cdot\vec{V})\,d^3x$

$$\Rightarrow \oint_S \vec{E}\cdot\underbrace{\hat{n}\,da}_{=d\vec{a}} = \underbrace{\int_V(\vec{\nabla}\cdot\vec{E})\,d^3x = \frac{1}{\epsilon_0}\int_V\rho(\vec{x})\,d^3x}$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{x})}{\epsilon_0}}$$

Consider point charge at origin, $\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}$

$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right) = \frac{q}{\epsilon_0}\cdot\delta(\vec{r}) \Rightarrow \rho(\vec{x}) = q\,\delta(\vec{r})$$ ✓

$$= 4\pi\delta(\vec{r})$$ ⟶ see supplement page

Divergence of $\frac{\hat{r}}{r^2}$: $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$

$$\underbrace{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)}_{\text{spherical}} = \frac{1}{r^2}\partial_r\left(r^2\cdot\frac{1}{r^2}\right) = \frac{1}{r^2}\cdot\partial_r(1) = \begin{cases} 0 & \text{for } r>0 \\ \frac{0}{0} & \text{at } r=0 \ \ !? \\ & \text{(not defined)} \end{cases}$$

By Gauss's Theorem: Consider sphere

$$\int_V \left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) d^3x = \oint_S \frac{\hat{r}}{r^2}\cdot\underbrace{d\vec{a}} = \oint_S \sin\theta\, d\theta\, d\phi = 4\pi$$

$$d\vec{a} = r^2\sin\theta\, d\theta\, d\phi\, \hat{r}$$

$\Rightarrow$ must have $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\,\delta(\vec{r})$!!

for Divergence Thm to be satisfied