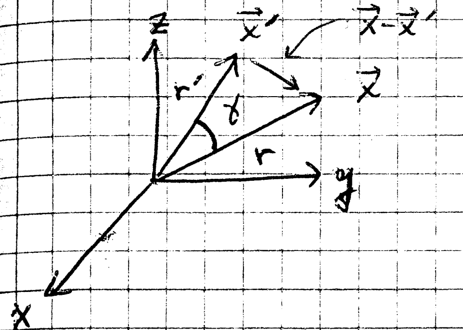
Point charge

An important expansion in terms of Legendre Polynomials is that of the potential due to a point charge at $\vec{x}'$:



$$\Phi(\vec{x}) \propto \frac{1}{|\vec{x}-\vec{x}'|}$$

$\vec{x}$: observation point
$\vec{x}'$: source point

→ With no loss of generality, take $\vec{x}'$ along z-axis (can always rotate axes).
Now,
- potential satisfies Laplace equation, for all $\vec{x} \neq \vec{x}'$
- problem now possesses azimuthal symmetry (no ϕ-dependence)

↳ can be expanded according to: $\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$
(except at $\vec{x}=\vec{x}'$)

So we have:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\left[A_l r^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\gamma) \qquad (\text{except at } \vec{x}=\vec{x}')$$

→ Now, if $\vec{x}$ is on z-axis, $\cos\gamma = 1$

Expand $\frac{1}{|\vec{x}-\vec{x}'|}$ "normally":

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{[(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')]^{1/2}} = \frac{1}{[r^2-2rr'\cos\gamma+r'^2]^{1/2}} \underset{\text{on z-axis}}{=} \frac{1}{[r^2-2rr'+r'^2]^{1/2}}$$

$$= \frac{1}{[(r-r')^2]^{1/2}} = \frac{1}{|r-r'|}$$

$[r=|\vec{x}|,\ r'=|\vec{x}'|]$

- If $r>r'$:

$$\frac{1}{|r-r'|} = \frac{1}{r(1-\frac{r'}{r})} = \frac{1}{r}\cdot\left(1+\left(\frac{r'}{r}\right)+\left(\frac{r'}{r}\right)^2+\dots\right) = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l$$

- If $r'>r$

$$\frac{1}{|r-r'|} = \frac{1}{|r'-r|} = \frac{1}{r'(1-\frac{r}{r'})} = \frac{1}{r'}\left(1+\left(\frac{r}{r'}\right)+\left(\frac{r}{r'}\right)^2+\dots\right) = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l$$

⇒ Compact notation: (for $\vec{x}$ and $\vec{x}'$ ∥ + $\hat{z}$)

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l$$

$r_>$ ($r_<$) is larger (smaller) of $|\vec{x}|$ and $|\vec{x}'|$

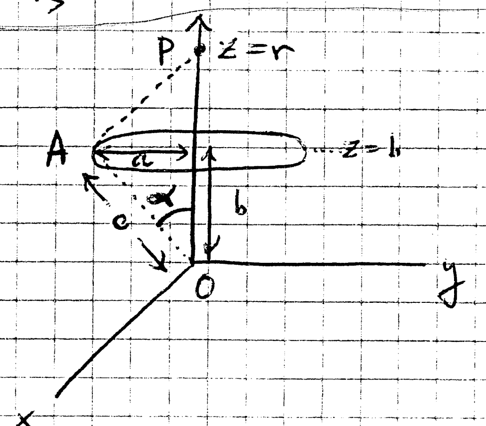
If $\vec{x}' \parallel \hat{z}$, but $\vec{x}$ arbitrary, we just need to multiply by $P_\ell(\cos\gamma)$:

b/c describe the angular dependence θ of the solution to $\nabla^2 \Phi = 0$!!

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\gamma)$$

$$= \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

Example: charged circular ring, charge q



By Law of Cosines:

$$(AP)^2 = r^2 + c^2 - 2cr\cos\alpha$$

$$\tan\alpha = \frac{a}{b}$$

Clearly, $\Phi(z=r) = \underbrace{\frac{q}{4\pi\epsilon_0}\frac{1}{(AP)}} = \frac{q}{4\pi\epsilon_0}\underbrace{\frac{1}{(r^2+c^2-2cr\cos\alpha)^{1/2}}}$ [analytic]

$$\left[\Phi = \int_{\text{ring}} \frac{dq}{4\pi\epsilon_0}\frac{1}{(AP)} = \frac{1}{(AP)}\frac{1}{4\pi\epsilon_0}\cdot\frac{q}{\cancel{2\pi a}}\cancel{(2\pi a)} = \frac{q}{4\pi\epsilon_0 (AP)}\right]$$

(constant)

But! This is just: $|\vec{OP}-\vec{OA}|^{-1}$

$$\frac{1}{|\vec{OP}-\vec{OA}|} = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{\ell=0}^{\infty}\left(\frac{c}{r}\right)^{\ell} P_\ell(\cos\alpha) & r > c \\[2ex] \dfrac{1}{c}\displaystyle\sum_{\ell=0}^{\infty}\left(\frac{r}{c}\right)^{\ell} P_\ell(\cos\alpha) & c > r \end{cases}$$

$|\vec{OP}| = r$

$|\vec{OA}| = c$ (r')

$$\Rightarrow \Phi(z=r) = \frac{q}{4\pi\epsilon_0}\cdot\begin{cases} \displaystyle\sum_{\ell=0}^{\infty}\frac{c^{\ell}}{r^{\ell+1}} P_\ell(\cos\alpha) & r > c \\[2ex] \displaystyle\sum_{\ell=0}^{\infty}\frac{r^{\ell}}{c^{\ell+1}} P_\ell(\cos\alpha) & c > r \end{cases}$$

This is for special case of P on z-axis, so for P anywhere in space, (r,θ):

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\cdot\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\alpha)\cdot P_\ell(\cos\theta)$$

$r_>$ $(r_<)$ is larger (smaller) of r and c.

Associated Legendre Functions and Spherical Harmonics

So far, azimuthal symmetry. Unless range of θ is restricted, e.g., $\alpha < \theta < \beta$, (e.g., Jackson 3.4), solutions involve only the "ordinary" Legendre polynomials, $P_\ell(x)$.

If don't have azimuthal symmetry, need restore $Q(\phi) = e^{\pm im\phi}$ in the solution: $\Phi = \frac{U(r)}{r} P(\theta) Q(\phi)$

Also, now the solution for the $P(\theta)$:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \underbrace{\frac{m^2}{1-x^2}}_{m \neq 0 \text{ if no azimuthal symmetry!}}\right] P = 0 \quad , x = \cos\theta$$

Solutions are so-called Associated Legendre Functions: $P_\ell^m(x)$ on interval $-1 \le x \le 1$
$\ell = 0$, positive integer
$m: -\ell, -\ell+1, \ldots, \ell-1, \ell$ (integers)

For $m > 0$: $P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$

can be shown: $P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$

Note: Rodrigues' formula for $P_\ell(x)$ is: $P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

~~so we can write:~~

$$m > 0: \quad P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

For fixed m,

$$\int_{-1}^{+1} P_{\ell'}^m(x) P_\ell^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell'\ell} \quad (*) \quad \text{(i.e., orthogonal)}$$

Solution of $\vec{\nabla}^2 \Phi(\vec{x}) = 0$ in spherical coordinates was cast in separation of variables as:

$$\Phi(r,\theta,\phi) = \frac{U(r)}{r} \cdot \underbrace{P(\theta)}_{\underbrace{P_\ell^m(\cos\theta)}_{\substack{\text{orthogonal} \\ \text{on } -1 \le \cos\theta \le 1}}} \cdot \underbrace{Q(\phi)}_{\underbrace{e^{\pm im\phi}}_{\text{orthogonal on } 0 \le \phi \le 2\pi}}$$

$\Rightarrow$ product $P_\ell^m(\cos\theta) \cdot e^{\pm im\phi}$ forms complete orthogonal set on surface of unit sphere
θ, ϕ: product separated

Given $(*)$ and fact that:

$$\int_0^{2\pi} Q_m^* Q_{m'}\, d\phi = \int_0^{2\pi} e^{-im\phi} e^{im'\phi}\, d\phi = \int_0^{2\pi} d\phi\, e^{i(m'-m)\phi} = 2\pi\delta_{m'm}$$

[m, m' integers]

$$\Rightarrow \int_{-1}^{+1} d(\cos\theta)\, P_\ell^m(\cos\theta)\, P_{\ell'}^m(\cos\theta) \int_0^{2\pi} d\phi\, e^{-im\phi} e^{im'\phi} = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell'\ell} \cdot 2\pi\,\delta_{m'm}$$ (68)

$\Rightarrow$ If we define: $$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\cdot\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\cdot e^{im\phi}$$

such a function is orthonormal on $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. "Spherical Harmonics"

i.e., $$\int d\Omega\, Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell,m}(\theta,\phi) = \delta_{\ell'\ell}\,\delta_{m'm}$$

First few:

$\ell = 0$: $m=0$: $Y_{00}(\theta,\phi) = \dfrac{1}{\sqrt{4\pi}}$

$\ell = 1$:
$m=+1$ $\quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}}(\sin\theta)\, e^{i\phi}$
$m=0$ $\quad Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$
$m=-1$ $\quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}(\sin\theta)\, e^{-i\phi}$

etc., ...

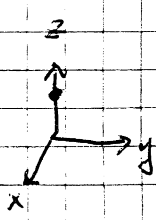
Note: $Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell,m}(\theta,\phi)$ (see 68b)

Now, as the $Y_{\ell,m}(\theta,\phi)$ comprise a set of orthonormal functions on: $0\le\theta\le\pi$, $0\le\phi\le 2\pi$, an arbitrary function $f(\theta,\phi)$ can be expanded in spherical harmonics as:

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$$

$$A_{\ell m} = \int d\Omega\, Y^*_{\ell,m}(\theta,\phi)\, f(\theta,\phi)$$ (see 68b)

Special case: Note that if: $\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow P_\ell(1) = 1\ \forall\,\ell$



we get: + @ same point for all $\phi \Rightarrow$ must have no ϕ-dependence, so $m=0$!

$$\underbrace{f(\theta,\phi)\Big|_{\theta=0}}_{\text{function we want to expand for } \theta=0} = \sum_{\ell=0}^{\infty} A_{\ell,0}\, Y_{\ell 0}(\theta)\Big|_{\theta=0}$$

$$= \sum_{\ell=0}^{\infty} A_{\ell,0}\sqrt{\frac{2\ell+1}{4\pi}}\cdot 1$$

where: $$A_{\ell,0} = \int d\Omega\, Y^*_{\ell,0}(\theta)\, f(\theta,\phi)$$ $\qquad Y^*_{\ell,0}(\theta,\phi) = (-1)^0\cdot Y_{\ell,0}(\theta,\phi)$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\int d\Omega\, P_\ell(\cos\theta)\, f(\theta,\phi)$$

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\cdot e^{im\phi}$$

Then: if $m \to -m$

$$Y_{l,-m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\cdot P_l^{-m}(\cos\theta)\cdot e^{-im\phi}$$

But, $P_l^{-m}(\cos\theta) = (-1)^m \dfrac{(l-m)!}{(l+m)!}\cdot P_l^m(\cos\theta)\, e^{-im\phi}$

$$\Rightarrow Y_{l,-m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\cdot(-1)^m\frac{(l-m)!}{(l+m)!}\,P_l^m(\cos\theta)\,e^{-im\phi}$$

$$= (-1)^m\sqrt{\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}}\;P_l^m(\cos\theta)\cdot\left(e^{im\phi}\right)^*$$

$$= (-1)^m\cdot Y^*_{l,m}(\theta,\phi) \checkmark$$

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} A_{lm}Y_{lm}(\theta,\phi)$$

$$\int Y^*_{l'm'}(\theta,\phi)\, f(\theta,\phi)\, d\Omega =$$

$$\int d\Omega \underbrace{\left[\sum_{l,m} A_{lm}Y_{lm}(\theta,\phi)\right] Y^*_{l'm'}(\theta,\phi)}$$

$$= A_{lm}\cdot\delta_{ll'}\,\delta_{mm'}$$

$$= A_{l'm'}$$