General solution to $\vec{\nabla}^2\Phi(\vec{x})=0$ can now be written as:

$$\Phi(r,\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\left[A_{\ell m}r^{\ell}+\frac{B_{\ell m}}{r^{\ell+1}}\right]Y_{\ell,m}(\theta,\phi)$$

If Φ specified on a spherical surface, $A_{\ell m}$ and $B_{\ell m}$ determined by:
e.g., $f(\theta,\phi)$

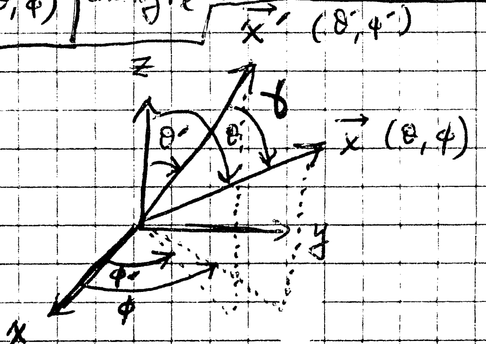
• evaluating $\Phi(r=R,\theta,\phi)$; and then <u>comparing with</u>: (expansion of $f(\theta,\phi)$)

$$\bullet\quad f(\theta,\phi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}C_{\ell m}Y_{\ell m}(\theta,\phi)$$

$$C_{\ell m}=\int d\Omega\, Y^{*}_{\ell m}(\theta,\phi)\, f(\theta,\phi)$$

Note: if $m=0$, $Y_{\ell 0}=\sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$, just only P_ℓ will suffice for $m=0$!

<u>Addition Theorem:</u>



Result from mathematics: if have $\vec{x}$ and $\vec{x}'$

$$\cos\gamma=\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|\,|\vec{x}'|}$$

$$P_\ell(\cos\gamma)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

→ expresses Legendre polynomial of order ℓ in the angle γ in terms of the products of spherical harmonics in (θ,ϕ) and (θ',ϕ').

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

<u>Proof:</u> see Jackson §3.6. (better Arfken §12.8)

Note if $\gamma=0\rightarrow\cos\gamma=1$, $\vec{x}\parallel\vec{x}'\Rightarrow(\theta,\phi)=(\theta',\phi')$

Recalling $P_\ell(1)=1\ \forall\,\ell$:

$$1=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\theta,\phi)\,Y_{\ell m}(\theta,\phi)=\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\left|Y_{\ell m}(\theta,\phi)\right|^2$$

"sum rule"

One useful "application" of the addition theorem:

Recall, potential at $\vec{x}$ due to point charge at $\vec{x}'$:

$$\frac{1}{|\vec{x}-\vec{x}'|}=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}\underbrace{P_\ell(\cos\gamma)}$$

$$=\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}\frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}Y^{*}_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)=4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}\cdot Y^{*}_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)\right]$$

$r_>$ ($r_<$) is larger (smaller) of $|\vec{x}|$ and $|\vec{x}'|$

$\Rightarrow$ Gives the potential due to a point charge, $\frac{1}{|\vec{x}-\vec{x}'|}$, in completely factorized form in the coordinates of $\vec{x}$ and $\vec{x}'$.

~~Now we will put the Y_{lm} to rest for a while until we discuss~~ multipoles.

Laplace Equation in Cylindrical Coordinates : (ρ, ϕ, z)

$$\vec{\nabla}^2 \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\hookrightarrow \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Usual separation of variables procedure; $\Phi(\rho, \phi, z) = R(\rho)\cdot Q(\phi)\cdot f(z)$

$$\Rightarrow \frac{1}{\rho} Q f \frac{dR}{d\rho} + Q f \frac{d^2R}{d\rho^2} + \frac{1}{\rho^2} R f \frac{d^2Q}{d\phi^2} + RQ \frac{d^2 f}{dz^2} = 0$$

$$\left[\times \frac{1}{RQf}\right] \Rightarrow \frac{1}{\rho}\frac{1}{R}\frac{dR}{d\rho} + \frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2Q}{d\phi^2} + \underbrace{\frac{1}{f}\frac{d^2 f}{dz^2}}_{z\text{-dependence isolated}} = 0$$

$$\boxed{\frac{1}{f}\frac{d^2 f}{dz^2} \equiv k^2}$$ convention, revisit later...

$$\frac{d^2 f}{dz^2} - k^2 f = 0$$

$$\Rightarrow \left(\frac{1}{\rho R}\frac{dR}{d\rho} + \frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{\rho^2}\frac{1}{Q}\frac{d^2Q}{d\phi^2} + k^2 = 0\right) \times \rho^2$$

$$\Rightarrow \frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{1}{Q}\frac{d^2Q}{d\phi^2} + \rho^2 k^2 = 0$$

$$\frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \rho^2 k^2 + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\phi\text{-dependence isolated}} = 0$$

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} \equiv -\nu^2$$

$$\frac{d^2Q}{d\phi^2} + \nu^2 Q = 0$$

$$\Rightarrow \frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \rho^2 k^2 - \nu^2 = 0$$

$$\left[\times \frac{R}{\rho^2}\right]: \quad \frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + R\left(k^2 - \frac{\nu^2}{\rho^2}\right) = 0$$

- $f(z) = e^{\pm kz}$ ($\sinh(kz)$, $\cosh(kz)$), take $k \geq 0$
- $Q(\phi) = e^{\pm i\nu\phi}$ ($\cos\nu\phi$, $\sin\nu\phi$), ν = integer (for Q to be single-valued) (just like before in the spherical case)

For "radial equation": let $x \equiv k\rho \Rightarrow dx = k\,d\rho$,

$$\frac{d}{d\rho} = \frac{d}{dx}\frac{dx}{d\rho} = k\frac{d}{dx}\ ; \quad \frac{d^2}{d\rho^2} = \frac{d}{d\rho}\left(k\frac{d}{dx}\right) = k\cdot k\frac{d}{dx}\frac{d}{dx} = k^2\frac{d^2}{dx^2}$$

$$\Rightarrow k^2\frac{d^2R}{dx^2} + \frac{k}{x}k\frac{dR}{dx} + R\left(k^2 - \frac{\nu^2 k^2}{x^2}\right) = 0$$

$$\left[x\frac{1}{k^2}\right]: \quad \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R\left(1 - \frac{\nu^2}{x^2}\right) = 0 \qquad \text{"Bessel Equation"}$$

Solutions: Bessel functions of order ν.

Note: Recall from your undergraduate differential equations class: This is a second-order differential equation $\Rightarrow$ must be 2 linearly independent solutions.

Two (possible) solutions: Bessel functions of the first kind of order $\pm\nu$.

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k}$$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{2k}$$

Recall: gamma function
$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$
Int. by parts: $\Gamma(z+1) = z\Gamma(z)$
$\Gamma(1) = 1 \Rightarrow \Gamma(n) = (n-1)!$

If $\nu \neq$ integer, (e.g., $\frac{1}{2}$), these constitute two linearly independent solutions.

If $\nu =$ integer, straightforward to show explicitly that:

$$J_{-\nu}(x) = (-1)^\nu J_\nu(x) \Rightarrow \underline{\text{NOT}} \text{ linearly independent}$$

Need to find another linearly-independent solution if $\nu =$ integer.

If half-integer argument:
$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$
$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-4)^n n!}{(2n)!}\sqrt{\pi}$

Customary: even if $\nu \neq$ integer, to replace $J_{\pm\nu}(x)$ with $J_\nu(x)$ and $N_\nu(x)$:

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad \text{"Neumann function"; or "Bessel function of the second kind"}$$

For $\nu \neq$ integer, N_ν clearly indep't of J_ν
For $\nu =$ integer, $\rightarrow \frac{0}{0} \rightarrow$ L'Hopital's Rule $\rightarrow$ indep't of J_ν!

Hankel functions ("Bessel functions of the third kind"), defined as:

$$\left.\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iN_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iN_\nu(x) \end{aligned}\right\} \text{ form fundamental set of solutions to the Bessel equation}$$

J_ν, N_ν, $H_\nu^{(1)}$, and $H_\nu^{(2)}$ all satisfy the recursion relations:

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x}\Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\frac{d\Omega_\nu(x)}{dx}$$

$$\Omega_\nu \in \left\{J_\nu, N_\nu, H_\nu^{(1)}, H_\nu^{(2)}\right\}$$

Note: for $\underset{=k\rho}{x} \ll 1$:

$$J_\nu(x) \rightarrow \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} + O(x^{\nu+2})$$

$$N_\nu(x) \rightarrow \begin{cases} \frac{2}{\pi}\left[\ln\frac{x}{2} + 0.5772\ldots\right] & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

$\uparrow$ blows up at $x = 0$!

e.g., $\nu = \frac{1}{2}$:

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{2k}$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{2k}$$

---

e.g., $\nu = 1$:

$$J_1(x) = \left(\frac{x}{2}\right)^{1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+2)} \left(\frac{x}{2}\right)^{2k}$$

$$J_{-1}(x) = \left(\frac{x}{2}\right)^{-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k)} \left(\frac{x}{2}\right)^{2k}$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma(7/2)}\left(\frac{x}{2}\right)^4 + \dots\right]$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \left[\frac{1}{\Gamma(1/2)} - \frac{1}{\Gamma(3/2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma(5/2)}\left(\frac{x}{2}\right)^4 + \dots\right]$$

Clearly, $J_{1/2}(x) \not\propto J_{-1/2}(x)$

$$J_1(x) = \left(\frac{x}{2}\right) \left[\frac{1}{\Gamma(2)} - \frac{1}{\Gamma(3)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma(4)}\left(\frac{x}{2}\right)^4 + \dots\right]$$

$$J_{-1}(x) = \left(\frac{x}{2}\right)^{-1} \left[\frac{1}{\Gamma(0)} - \frac{1}{\Gamma(1)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma(2)}\left(\frac{x}{2}\right)^4 + \dots\right]$$

not defined!

$$= \left(\frac{x}{2}\right)^{-1} \left[- \frac{1}{\Gamma(1)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\,\Gamma(2)}\left(\frac{x}{2}\right)^4 + \dots\right]$$

$$\Rightarrow J_1(x) = \left[\frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^3 + \frac{1}{2\cdot 3!}\left(\frac{x}{2}\right)^5 + \dots\right]$$

$$J_{-1}(x) = \left[-\frac{x}{2} + \frac{1}{2}\cdot\left(\frac{x}{2}\right)^3 + \dots\right]$$

$$= (-1)\cdot J_1(x)$$