(72)

and for $X >> 1, \nu$: ($x = k\rho$)
$$\begin{cases} J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \\ N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \end{cases}$$

Asymptotic forms ~~suggest~~ Bessel functions have ∞ number of roots:

e.g., in asymptotic limit, $x >> 1, \nu$, roots of $J_\nu(x)$: $x - \frac{\nu\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

In general: $J_\nu(x_{\nu n}) \equiv 0$, $x_{\nu n}$ are roots

$\nu = 0$: $x_{0n} = 2.405, 5.520, 8.654, \ldots$

$\nu = 1$: $x_{1n} = 3.832, 7.016, 10.173, \ldots$

$\nu = 2$: $x_{2n} = 5.136, 8.417, 11.620, \ldots$

So, to summarize, now know complete solution to $\vec{\nabla}^2 \Phi = 0$ in cylindrical coordinates:

$$\Phi(\rho, \phi, z) = R(\rho) \cdot Q(\phi) \cdot f(z)$$

$f(z) = e^{\pm kz}$, or linear combination, e.g., $\sinh(kz)$, $\cosh(kz)$; $k \geq 0$

$Q(\phi) = e^{\pm i\nu\phi}$, or linear combination of any:

e.g., $e^{+i\nu\phi} + e^{-i\nu\phi} \propto \cos\nu\phi$, $e^{i\nu\phi} - e^{-i\nu\phi} \propto \sin\nu\phi$ } ν integer ≥ 0

$R(\rho) = J_\nu(x), N_\nu(x) \Rightarrow J_\nu(k\rho), N_\nu(k\rho)$

Note: Bessel functions of first kind form an orthogonal, complete set of functions; with basis functions: $\sqrt{\rho}\, J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)$ fixed ν ($\nu \geq 0$), $n = 1, 2, 3, \ldots$ on interval $0 \leq \rho \leq a$; $x_{\nu n}$: roots! of J_ν

Orthogonality condition:
$$\int_0^a \rho J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \delta_{n'n}$$

Arbitrary function of ρ on the interval $0 \leq \rho \leq a$ can be expanded as:
$$f(\rho) = \sum_{n=1}^{\infty} A_{\nu n} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)$$
"Fourier-Bessel series", $\nu \geq 0$

$$A_{\nu n} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} \int_0^a \rho f(\rho) J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) d\rho$$ (see 72b)

$\Rightarrow$ particularly useful for functions that vanish at $\rho = a$ (e.g., Dirichlet boundary-value problems)!

take to be 0 at root of Bessel function

Fourier - Bessel Series:

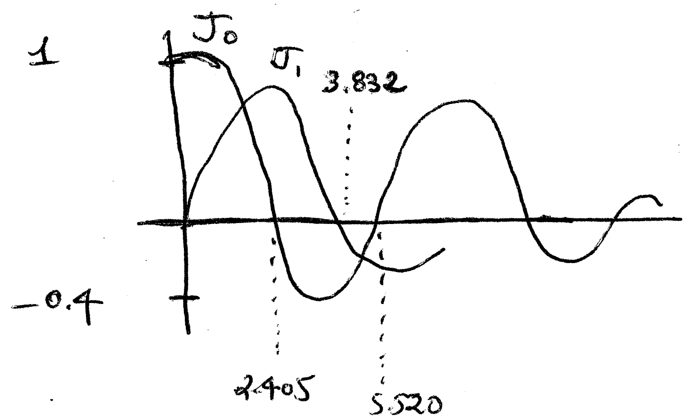
Assumption: start with: $f(\rho) = \sum_{m=1}^{\infty} A_{\nu m} J_\nu\left(\frac{x_{\nu m}\rho}{a}\right)$

$$\Rightarrow \rho \cdot f(\rho) = \sum_{m=1}^{\infty} \rho A_{\nu m} J_\nu\left(\frac{x_{\nu m}\rho}{a}\right)$$

$$\Rightarrow \int_0^a d\rho\, J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)\cdot \rho \cdot f(\rho) = \int_0^a d\rho \cdot J_\nu\left(\frac{x_{\nu n}\rho}{a}\right)\left[\sum_{m=1}^{\infty} \rho A_{\nu m} J_\nu\left(\frac{x_{\nu m}\rho}{a}\right)\right]$$

$$= A_{\nu n}\cdot \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu n})\right]^2 \qquad \text{(from orthogonality)}$$

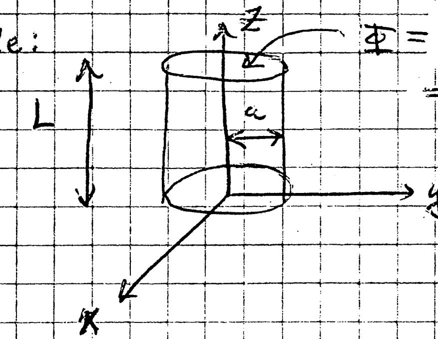
$$\Rightarrow A_{\nu n} = \frac{2}{a^2}\frac{1}{\left[J_{\nu+1}(x_{\nu n})\right]^2}\int_0^a d\rho \cdot \rho \cdot f(\rho)\cdot J_\nu\left(\frac{x_{\nu n}\rho}{a}\right) \checkmark$$



realistic, not asymptotic limiting forms.

Boundary-Value Problems in Cylindrical Coordinates

Example:



$\Phi = V(\rho, \phi)$ (top lid)

$= 0$ everywhere else!

Find Φ everywhere inside the cylinder.

Know: $\Phi = R(\rho)\, Q(\phi)\, f(z)$

① need $\Phi = 0$ at $z = 0$

$f(z) = \sinh(kz)$

② need Φ to be single-valued in ϕ

$\Rightarrow$ in $Q(\phi) = \{e^{\pm i\nu\phi}\}$ need $\nu = m =$ integer

$\Rightarrow$ take $Q(\phi) = A \sin m\phi + B \cos m\phi$

③ can't have $\Phi \to \infty$ at $\rho = 0$, so in $R(\rho)$

$N_m(k\rho)$ is not acceptable!

$R(\rho) = J_m(k\rho)$, $m =$ integer only!

$$\Rightarrow \Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} J_m(k\rho)\left[A \sin m\phi + B \cos m\phi\right] \sinh(kz)$$

Finally, need $\Phi = 0$ for $\rho = a$. $J_m(k\rho)$ has infinite number of roots:

if $k_{mn} a = x_{mn}$ (x_{mn} are the $n = 1, 2, 3, \ldots$ roots of J_m)

$$J_m(k_{mn} a) = 0$$

$$\Rightarrow \Phi(\rho, \phi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn} z) \cdot \left[A_{mn} \sin m\phi + B_{mn} \cos m\phi\right]$$

Boundary condition is at $z = L$:

(double Fourier + Fourier–Bessel series!)

$$V(\rho, \phi) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}\rho) \sinh(k_{mn} L) \left[A_{mn} \sin m\phi + B_{mn} \cos m\phi\right]$$

To determine A_{mn}:

Multiply by: $\rho \cdot J_p(k_{pq}\rho) \cdot \sin(p\phi)$, $\begin{cases} p: \text{ integer} \geq 0 & \text{(analogous to } m) \\ q: \text{ integer} \geq 1 & \text{(to } n) \end{cases}$

$$\int_0^{2\pi} d\phi \int_0^a d\rho\, \rho\, J_p(k_{pq}\rho) \sin(p\phi) \cdot V(\rho, \phi) = \int_0^{2\pi} d\phi \int_0^a d\rho \cdot \rho\, J_p(k_{pq}\rho) \sin(p\phi) \cdot \sum_{m,n} \Big[J_m(k_{mn}\rho) \cdot \underbrace{\sinh(k_{mn} L)}_{\text{just a number!}} A_{mn} \sin(m\phi) \Big]$$

no $\cos(m\phi)$ as $\int_0^{2\pi} d\phi \sin(p\phi) \cos(m\phi) = 0 \;\forall\; p, m$