$$\Rightarrow \int_0^{2\pi} d\phi \int_0^a d\rho \cdot \rho \cdot V(\rho,\phi)\, J_p(k_{pq}\rho) \cdot \sin(p\phi) =$$

$$\sum_{m,n} \Bigg[\underbrace{\int_0^{2\pi} d\phi \, \sin(p\phi)\sin(m\phi)}_{= \pi \cdot \delta_{mp}} \int_0^a d\rho \, \rho \, J_p(k_{pq}\rho) J_m(k_{mn}\rho) \Bigg] \cdot A_{mn} \sinh(k_{mn} L)$$

$= \pi \cdot \delta_{mp} \Rightarrow m = p$ only (eliminate $\sum_m$)

$$= \sum_n \pi \cdot \Bigg[\underbrace{\int_0^a d\rho \cdot \rho \cdot J_p(k_{pq}\rho) J_p(k_{pn}\rho)}_{= \frac{a^2}{2}\left[J_{p+1}(x_{pq})\right]^2 \delta_{qn}} \Bigg] A_{pn} \cdot \sinh(k_{pn} L)$$

$= \frac{a^2}{2}\left[J_{p+1}(x_{pq})\right]^2 \delta_{qn} \Rightarrow n = q$ only, eliminate $\sum_n$

$$\Rightarrow \frac{\pi a^2}{2} \cdot \left[J_{p+1}(x_{pq})\right]^2 \cdot A_{pq} \cdot \sinh(k_{pq} L) = \int_0^{2\pi} d\phi \int_0^a d\rho \cdot \rho \cdot V(\rho,\phi)\, J_p(k_{pq}\rho) \cdot \sin(p\phi)$$

$$\boxed{A_{pq} = \frac{2}{\sinh(k_{pq}L)\, \pi a^2 \left[J_{p+1}(x_{pq})\right]^2} \cdot \int_0^{2\pi} d\phi \int_0^a d\rho \cdot \rho \cdot V(\rho,\phi)\, J_p(k_{pq}\rho) \sin(p\phi)}$$

Similarly, would find: [multiply by $\rho \cdot J_p(k_{pq}\rho) \cdot \cos(p\phi)$]

$$\boxed{B_{pq} = \frac{2}{\sinh[k_{pq}L]\, \pi a^2 \left[J_{p+1}(x_{pq})\right]^2} \cdot \int_0^{2\pi} d\phi \int_0^a d\rho \cdot \rho \, V(\rho,\phi)\, J_p(k_{pq}\rho) \cos(p\phi)}$$

<u>Recall</u>: In postulating separation of variables, $\Phi(\rho,\phi,z) = R(\rho)\,Q(\phi)\,f(z)$

we took: $\frac{1}{f}\frac{d^2 f}{dz^2} \equiv k^2 \implies f(z) \propto e^{\pm kz},\ \sinh(kz),\ \cosh(kz)$

Could instead have defined:

$$\frac{1}{f}\frac{d^2 f}{dz^2} \equiv -k^2 \implies f(z) \propto \sin(kz),\ \cos(kz)$$

Then, with $\frac{1}{Q}\frac{d^2 Q}{d\phi^2} \equiv -\nu^2$ as before, would have for ρ equation: (see p. 70)

$$\frac{\rho}{R}\frac{dR}{d\rho} + \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} - \rho^2 k^2 - \nu^2 = 0$$

as before!

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - k^2 R - \nu^2\frac{R}{\rho^2} = 0 \implies \text{take } k\rho \equiv x \implies dx = k\,d\rho \implies \frac{d}{dx} = \frac{1}{k}\frac{d}{d\rho}$$

$$\frac{d^2}{dx^2} = \frac{1}{k^2}\frac{d^2}{d\rho^2}$$

$$k^2\frac{d^2R}{dx^2} + \frac{k}{x}\cdot k\frac{d}{dx}R - k^2 R - \nu^2 R\cdot\frac{k^2}{x^2} = 0$$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - R - R\frac{\nu^2}{x^2} = 0 \implies \underline{\underline{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - R\left(1+\frac{\nu^2}{x^2}\right) = 0}}$$

$$\left[\text{c.f. } \frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + R\left(1-\frac{\nu^2}{x^2}\right) = 0\right.$$

<u>Solutions</u>: "Modified Bessel Functions": essentially, took $k \to ik$ ($k^2 \to -k^2$), $x \to ix$ in argument of Bessel functions

$$\left.\begin{aligned} I_\nu(x) &= i^{-\nu} J_\nu(ix) \\ K_\nu(x) &= \frac{\pi}{2}\cdot i^{\nu+1} H_\nu^{(1)}(ix) \end{aligned}\right\} \text{ real functions for real } x \text{ and } \nu$$

<u>For $\nu \geq 0$</u>:

$$x \ll 1: \begin{cases} I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \to 0 \text{ as } x \to 0 \quad \text{(well-behaved)} \\ K_\nu(x) \to \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\ldots\right] & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu & \ldots \end{cases} \end{cases}$$

$$x \gg 1, \nu: \begin{cases} I_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^x\left[1 + O\left(\frac{1}{x}\right)\right] \\ K_\nu(x) \to \sqrt{\frac{\pi}{2x}}\, e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right] \to 0 \text{ as } x \to \infty \quad \text{(well-behaved)} \end{cases}$$

Motivation: Green function $G(x, x')$ for soln of Poisson Eqn in presence of b.c.'s.
In rectangular/cylindrical/spherical, some solutions are separable in the coordinates.
$\Rightarrow$ want to express $G(\vec{x}, \vec{x}')$ as product of these separable coordinates.

$\boxed{\text{Expansion of Green Functions in Cylindrical Coordinates}}$

Goal: Write expansion for potential of a point charge in cylindrical coordinates (ρ, ϕ, z).

In rectangular coordinates, recall: $\vec{\nabla}^2_{xyz} G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')$ [general definition of $G(\vec{x}, \vec{x}')$]

$$= \delta(x - x')\,\delta(y - y')\,\delta(z - z')$$

$$\Rightarrow \int \delta(\vec{x} - \vec{x}')\, dx\,dy\,dz = 1 \text{ if } \vec{x} = \vec{x}'$$

In cylindrical coordinates, need:

$$\int \delta(\vec{x} - \vec{x}')\, \rho\, d\rho\, d\phi\, dz = 1 \text{ if } \vec{x} = \vec{x}'$$

$$\Rightarrow \delta(\vec{x} - \vec{x}') = \frac{1}{\rho}\,\delta(\rho - \rho') \cdot \delta(\phi - \phi') \cdot \delta(z - z')$$

Write expansion for $\delta(\phi - \phi')$ and $\delta(z - z')$ in terms of orthonormal functions:

Recall: In general, $\delta(u - u') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(u-u')}\, dk$, for u on the interval $(-\infty, \infty)$ so we can write for $\delta(z - z')$:

where z defined on $(-\infty, \infty)$

$$\Rightarrow \delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(z-z')}\, dk$$

odd function in k, and $\int_{-L}^{L} dx\, f(x) = 0$ as $f(x) = -f(-x)$ (odd)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \left[\cos[k(z-z')] + i \sin[k(z-z')]\right] = \frac{1}{\cancel{2}\pi} \int_0^{\infty} dk \cdot \underbrace{\cancel{2} \cdot \cos[k(z-z')]}_{\text{even}} = \frac{1}{\pi} \int_0^{\infty} dk \cos[k(z-z')]$$

and recall that: $\sin(m\alpha)$ and $\cos(m\alpha)$ constitute an orthonormal basis set of functions on $\alpha \in [0, 2\pi]$, so we can expand $\delta(\phi - \phi')$ as:

$$\delta(\phi - \phi') = \sum_{m=-\infty}^{\infty} C_m e^{im(\phi - \phi')}$$

Recall: If $f(\xi) = \sum_n a_n u_n(\xi)$ on $[a, b]$:

$$a_n = \int_a^b d\xi\, u_n^*(\xi)\, f(\xi)$$

Find the C_m:

$$\int_0^{2\pi} \left[e^{im'(\phi - \phi')}\right]^* \delta(\phi - \phi')\, d(\phi - \phi') = \int_0^{2\pi} d(\phi - \phi') \left[e^{im'(\phi - \phi')}\right]^* \cdot \sum_{m=-\infty}^{\infty} C_m e^{im(\phi - \phi')}$$

$$\Rightarrow 1 = \int_0^{2\pi} du\, e^{-im'u} \sum_{m=-\infty}^{\infty} C_m e^{imu}$$

$$= \int_0^{2\pi} du \left(\cos(m'u) - i \sin(m'u)\right) \left[\sum_{m=-\infty}^{\infty} C_m \left[\cos(mu) + i \sin(mu)\right]\right]$$

$$= C_{m'} \left[\pi + \pi\right] \Rightarrow C_{m'} = \frac{1}{2\pi}$$

$$\Rightarrow \delta(\phi - \phi') = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi - \phi')}$$

using:

$$\int_0^{2\pi} dx \, \underset{\sin}{\cos}(nx)\, \underset{\sin}{\cos}(mx) = \pi\, \delta_{mn}$$

$$\int_0^{2\pi} dx \cos(mx) \sin(nx) = 0 \;\; \forall\, m, n$$