$$\Rightarrow \delta(\vec{x}-\vec{x}') = \frac{1}{(2\pi)(\pi)}\int_0 dk \cos[k(z-z')] \cdot \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \cdot \frac{1}{\rho}\cdot \delta(\rho-\rho')$$

$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\cdot\frac{1}{\rho}\,\delta(\rho-\rho')$$

We can similarly expand $G(\vec{x},\vec{x}')$ as: in the ϕ and z coordinates as:

$$G(\vec{x}-\vec{x}') = G(\rho,\rho',\phi,\phi',z,z') =$$

$$\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\cdot \underbrace{g_m(\rho,\rho')}_{\text{"expansion coefficients"}}$$

Now, substitute this expansion of $G(\vec{x},\vec{x}')$ into $\vec{\nabla}^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$:

$\Rightarrow$ Integral over $\int dk$, so take $\vec{\nabla}^2$ inside integral:

$$\left[\vec{\nabla}^2\psi = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho\psi) + \frac{1}{\rho^2}\partial_\phi^2\psi + \partial_z^2\psi\right]$$

$$\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\,\vec{\nabla}^2\left\{e^{im(\phi-\phi')}\cdot\cos[k(z-z')]\cdot g_m(\rho,\rho')\right\}$$

$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot e^{im(\phi-\phi')}\cos[k(z-z')]\cdot\frac{\partial g_m}{\partial\rho}\right) + \frac{1}{\rho^2}\left(-m^2 e^{im(\phi-\phi')}\cos[k(z-z')]\,g_m(\rho,\rho')\right)\right.$$
$$\left. + -k^2\left(e^{im(\phi-\phi')}\cdot\cos[k(z-z')]\cdot g_m(\rho,\rho')\right)\right\}$$

$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\left\{\frac{1}{\rho}\, e^{im(\phi-\phi')}\cos[k(z-z')]\frac{\partial g_m}{\partial\rho} + e^{im(\phi-\phi')}\cos[k(z-z')]\frac{\partial^2 g}{\partial\rho^2}\right.$$
$$\left. - \frac{m^2}{\rho^2}e^{im(\phi-\phi')}\cos[k(z-z')]\,g_m(\rho,\rho') - k^2 e^{im(\phi-\phi')}\cos[k(z-z')]\,g_m(\rho,\rho')\right\}$$

this then

$$\hookrightarrow = -4\pi\,\delta(\vec{x}-\vec{x}') = -4\pi\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\,\frac{1}{\rho}\,\delta(\rho-\rho')$$

Demand that the integrals are equal: [noting all terms have $e^{im(\phi-\phi')}$ and $\cos[k(z-z')]$]

$$-4\pi\frac{1}{\rho}\delta(\rho-\rho') = \frac{1}{\rho}\frac{\partial g_m}{\partial\rho} + \frac{\partial^2 g_m}{\partial\rho^2} - \frac{m^2}{\rho^2}g_m - k^2 g_m$$

$$\Rightarrow \frac{\partial^2 g_m}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial g_m}{\partial\rho} - g_m\left(\frac{m^2}{\rho^2}+k^2\right) = \underbrace{-4\pi\frac{1}{\rho}\delta(\rho-\rho')}$$

$$\boxed{\begin{aligned}&\left[\frac{\partial^2 g_m}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial g_m}{\partial\rho} - g_m\left(\frac{m^2}{\rho^2}+k^2\right) = 0\right]\times\frac{1}{k^2}\\ &\frac{1}{k^2}\frac{\partial^2 g}{\partial\rho^2} + \frac{1}{k\rho}\cdot\frac{1}{k}\cdot\frac{\partial g_m}{\partial\rho} - g_m\left(\frac{m^2}{k^2\rho^2}+1\right) = 0\\ &\frac{\partial^2 g_m}{\partial x^2} + \frac{1}{x}\frac{\partial g_m}{\partial x} - g_m\left(\frac{m^2}{x^2}+1\right) = 0 \checkmark\end{aligned}}$$

if $\rho \neq \rho'$, this is the modified Bessel Equation (recall p. 72c)

$[x \Rightarrow k\rho]$

$dx \Rightarrow k\,d\rho \Rightarrow \frac{d}{dx} = \frac{1}{k}\frac{d}{d\rho}$ $\quad \frac{d^2}{dx^2} = \frac{1}{k^2}\frac{d^2}{d\rho^2}$

to modified Bessel Equation

Recall: solutions are: $I_m(x) = i^{-m} J_m(ix)$

$K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix)$ $\quad [x = k\rho]$

and for: $x \to 0 : I_m(x) \to 0$

$x \to \infty : K_m(x) \to 0$

$G(\vec{x}, \vec{x}')$ function of $\vec{x}, \vec{x}'$: considering ρ "variable", ρ' "fixed"

we must have (for a well-behaved solution for $\rho \to 0$ and $\rho \to \infty$): ρ' is fixed, so can have $\rho < \rho'$ or $\rho > \rho'$

$\rho' \to 0 \quad \rho' \to \infty$

write $g_m = A\left[I_m(k\rho_<) \cdot K_m(k\rho_>)\right]$ ($\rho_>$ is greater of ρ, ρ'; $\rho_<$ is smaller of ρ and ρ')

constant

Find A: ~~First note:~~ But $\frac{\partial^2 g_m}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial g_m}{\partial \rho} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial g_m}{\partial \rho}\right)$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial g_m}{\partial \rho}\right) - g_m\left(\frac{m^2}{\rho^2} + k^2\right) = -\frac{4\pi}{\rho}\delta(\rho - \rho')$$

Multiply by ρ:

$$\Rightarrow \frac{\partial}{\partial \rho}\left(\rho \frac{\partial g_m}{\partial \rho}\right) - \frac{g_m}{\rho}\left(\frac{m^2}{\rho^2} + k^2\right) = -4\pi\,\delta(\rho - \rho')$$

Integrate about $\rho' \pm \varepsilon$:

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho\, \frac{\partial}{\partial \rho}\left(\rho \frac{\partial g_m}{\partial \rho}\right) - \underbrace{\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho\, \frac{g_m}{\rho}\left(\frac{m^2}{\rho^2} + k^2\right)}_{\to 0 \text{ as } \varepsilon \to 0 \text{ as}} = -\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho\, 4\pi\,\delta(\rho - \rho')$$

$g_m \propto I \cdot K$ is "smooth/continuous" as is $\frac{1}{\rho^3}$

$$\Rightarrow \rho' \frac{\partial g_m}{\partial \rho}\bigg|_{\rho = \rho'+\varepsilon} - \rho' \frac{\partial g_m}{\partial \rho}\bigg|_{\rho = \rho'-\varepsilon} = -4\pi \Rightarrow$$ discontinuity in derivative at $\rho = \rho'$!

(just like in spherical radial Green function!)

But! We have proposed $g_m = A\, I_m \cdot K_m$ and:

$$\frac{\partial g_m}{\partial \rho}\bigg|_{\rho = \rho'+\varepsilon} - \frac{\partial g_m}{\partial \rho}\bigg|_{\rho = \rho'-\varepsilon} = \frac{\partial}{\partial \rho}\left[A \cdot I_m(k\rho') K_m(k\rho)\right] - \frac{\partial}{\partial \rho}\left[A\, I_m(k\rho) K_m(k\rho')\right]$$

$\begin{pmatrix}\rho_< = \rho' \\ \rho_> = \rho\end{pmatrix} \quad \begin{pmatrix}\rho_< = \rho \\ \rho_> = \rho'\end{pmatrix}$

$$= A\, I_m(k\rho') \frac{\partial}{\partial\rho}\left(K_m(k\rho)\right)\Big|_{\rho=\rho'} - A\, K_m(k\rho') \frac{\partial}{\partial\rho}\left(I_m(k\rho)\right)\Big|_{\rho=\rho'}$$

using: $\frac{d}{dx} = \frac{1}{k}\frac{d}{d\rho}$, $x = k\rho$

$$= Ak\left[I_m(x) \frac{\partial}{\partial x}\left(K_m(x)\right) - K_m(x) \frac{\partial}{\partial x}\left(I_m(x)\right) \right]\Bigg|_{\substack{x=k\rho=k\rho' \\ \rho=\rho'}}$$

$$= Ak\left[I_m K_m' - I_m' K_m \right]\Big|_{\rho=\rho'} = -\frac{4\pi}{\rho}\Big|_{\rho=\rho'}$$

[Value of Wronskian for all ρ.

But! This is just the Wronskian for the solutions, I & K, our second-order differential equation!

[modified Bessel Equation.

[Recall: Wronskian of two functions f and g is $W = fg' - f'g$ if f and g are linearly dependent, $W=0$

$\Rightarrow$ I & K are independent solutions!

Now, find A: (via trick)

In the limit of $x \to 0$, $\quad I_m \to \frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m \Rightarrow \frac{\partial}{\partial x} I_m = \frac{m x^{m-1}}{\Gamma(m+1)} \cdot \frac{1}{2^m}$

$$K_m \to \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m \Rightarrow \frac{\partial}{\partial x} K_m = \frac{\Gamma(m)}{2} \cdot 2^m \cdot (-m)\, \cancel{x^{-(m+1)}}$$

$$\Rightarrow Ak\left[\frac{1}{\Gamma(m+1)}\left(\frac{x}{2}\right)^m \cdot \frac{\Gamma(m)}{2} \cdot \frac{(-m)\,2^m}{x^{m+1}} - \frac{m}{\Gamma(m+1)} \frac{x^{m-1}}{2^m} \cdot \frac{\Gamma(m)}{2} \frac{2^m}{x^m} \right] \qquad \Gamma(m+1) = m!$$

$$= Ak\left[-\frac{1}{\cancel{m!}} \cdot \frac{x^m}{2^m} \cdot \frac{\cancel{(m-1)!\cdot m}\, 2^m}{2 \cdot x^{m+1}} - \frac{\cancel{m}}{\cancel{m!}} \frac{x^{m-1}}{\cancel{2^m}} \cdot \frac{\cancel{(m-1)!}}{\cancel{2}} \cdot \frac{\cancel{2^m}}{x^m} \right]$$

$$= Ak\left[-\frac{1}{x}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{x} \right] = -Ak\cdot\frac{1}{x}$$

Must be equal to: $-Ak\frac{1}{x} = -\frac{4\pi}{\rho} \Rightarrow -\cancel{Ak}\frac{1}{\cancel{k}\rho} = -\frac{4\pi}{\rho} \Rightarrow \underline{\underline{A = 4\pi}}$

(~~$\forall x$, so also valid for $x \to 0$~~)

Thus, the expansion of the Green function becomes:

$$G(\vec{x},\vec{x}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\cdot A\, I_m(k\rho_<)\cdot K_m(k\rho_>)$$

$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\cdot 4\pi\cdot I_m(k\rho_<)\cdot K_m(k\rho_>)$$

$$= \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\, I_m(k\rho_<)\, K_m(k\rho_>) \checkmark$$

[~~For point charge,~~ $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$]

$+ F(\vec{x},\vec{x}')$,

$\vec{\nabla}^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$, with $\vec{\nabla}^2 F(\vec{x},\vec{x}') = 0$

thus, $\frac{1}{|\vec{x}-\vec{x}'|}$ = expansion ...... as we equated $\vec{\nabla}^2 G \to -4\pi\,\delta(\vec{x}-\vec{x}')$