Multipole Expansion

Consider localized charge distribution, $\rho(\vec{x}')$. Non-zero only inside some sphere radius R centered on origin. (just separate space into charge and charge-free regions).

Outside the sphere, we know the solution for the potential in spherical coordinates!

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$\vec{x}$: observation point, $\vec{x}'$: sources which we are integrating over

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$\left[r_< = r',\ r_> = r\right.$ (as we are outside the sphere!!) $\left.\right]$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\int d^3x'\rho(\vec{x}')\frac{1}{(2l+1)}\frac{r'^l}{r^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$= \frac{1}{\varepsilon_0}\sum_{l,m}\underbrace{\left[\int d^3x'\rho(\vec{x}')(r')^l Y^*_{lm}(\theta',\phi')\right]}_{\equiv\, q_{lm}\ \text{"multipole moments"}}\frac{Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}$$

$l=0$: "monopole"
$l=1$: "dipole"
$l=2$: "quadrupole"
etc.

$\Phi(\vec{x})$ outside charged region; or if $\rho(\vec{x}')$ falls off faster than some $\frac{1}{r^n}$; otherwise, need to use $G(\vec{x},\vec{x}')$: Problem 4.7

$\left[Y_{l,-m}(\theta',\phi') = (-1)^m Y^*_{l,m}(\theta',\phi')\right]$

Note: for $m>0$

$$q_{l,-m} = \int d^3x'\rho(\vec{x}')(r')^l(-1)^m Y_{l,m}(\theta',\phi')$$

$$= (-1)^m q^*_{lm}\quad \text{(assuming } \rho(\vec{x}') \text{ is not complex-valued)}$$

Monopole: $l=0$, $\Rightarrow m=0$ only

$\boxed{\text{Physical Interpretation: Cartesian Coordinates!}}$

$$q_{00} = \int d^3x'\rho(\vec{x}')Y^*_{00}(\theta',\phi') = \int d^3x'\rho(\vec{x}')\frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}}q\ \checkmark$$

(total charge; $\Rightarrow$ monopole)

Dipole: $l=1 \Rightarrow m=-1,0,+1$

$$q_{11} = \int d^3x'\rho(\vec{x}')r'\cdot Y^*_{11}(\theta',\phi') = \int d^3x'\rho(\vec{x}')\cdot r'\cdot\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta' e^{-i\phi'}$$

$$= -\sqrt{\frac{3}{8\pi}}\int d^3x'\rho(\vec{x}')\Big[\underbrace{r'\sin\theta'\cos\phi'}_{=x'} - \underbrace{ir'\sin\theta'\sin\phi'}_{iy'}\Big] = -\sqrt{\frac{3}{8\pi}}\int d^3x'\rho(\vec{x}')(x'-iy')$$

$= -\sqrt{\frac{3}{8\pi}}\left(p_x - i p_y\right)$

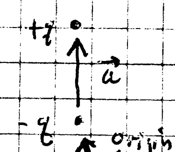
$$p_x \equiv \int d^3x' \rho(\vec{x}')\, x'$$
$$p_y \equiv \int d^3x' \rho(\vec{x}')\, y'$$
$$p_z \equiv \int d^3x' \rho(\vec{x}')\, z'$$
$$\underbrace{\vec{p} \equiv \int \vec{x}' \rho(\vec{x}')\, d^3x'}_{\text{"electric dipole moment"}}$$

Similarly,

$$q_{10} = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{1,-1} = (-1)^1 q_{11}^* = (-1)\cdot\left(-\sqrt{\frac{3}{8\pi}}\right)(p_x - i p_y)^*$$
$$= \sqrt{\frac{3}{8\pi}}\,(p_x + i p_y)$$

Simple example:



$$\rho(\vec{x}') = -q\,\delta(\vec{x}') + q\,\delta(\vec{x}' - \vec{a})$$
$$\vec{p} = \int [-q\,\delta(\vec{x}') + q\,\delta(\vec{x}' - \vec{a})]\cdot \vec{x}'\cdot d^3x'$$
$$= -q\cdot(\vec{0}) + q\cdot\vec{a}$$
$$= q\vec{a} \checkmark$$

$l = 2;\ m = -2, -1, 0, +1, +2:$

$$q_{22} = \int d^3x' \rho(\vec{x}')(r')^2\, Y_{22}^*(\theta', \phi')$$
$$= \int d^3x' \rho(\vec{x}')(r')^2 \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta' e^{-i2\phi'}$$
$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') \Big[\underbrace{r' \sin\theta' e^{-i\phi'}}_{= x' + i y'}\Big]^2$$
$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int d^3x' \rho(\vec{x}') [x' + i y']^2 \equiv \frac{1}{12}\sqrt{\frac{15}{2\pi}}\left[Q_{11} - 2i\,Q_{12} - Q_{22}\right]$$

$$Q_{ij} \equiv \int d^3x' \rho(\vec{x}')\left[3x_i' x_j' - (r')^2 \delta_{ij}\right] \qquad \begin{pmatrix} x_1' = x' \\ x_2' = y' \\ x_3' = z' \end{pmatrix}$$

$$\Rightarrow\ Q_{11} = \int d^3x' \rho(\vec{x}')\left[3x'^2 - (x'^2 + y'^2 + z'^2)\right]$$
$$Q_{22} = \int d^3x' \rho(\vec{x}')\left[3y'^2 - (x'^2 + y'^2 + z'^2)\right]$$

[q_{21}, q_{20} in Jackson p. 146 ...]

$$\Rightarrow\ Q_{11} - Q_{22} = 3\int d^3x' \rho(\vec{x}')\,(x'^2 - y'^2) \checkmark$$
$$Q_{12} = \int d^3x' \rho(\vec{x}')\cdot 3\cdot x' y' \checkmark$$

Q_{ij}: "quadrupole moment tensor"; General expansion of $\Phi(\vec{x})$ in rectangular:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots\right]$$

What about the electric field? — will use later...

$$\Phi(\vec{x}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{q_{lm} Y_{lm}(\theta,\phi)}{(2l+1) r^{l+1}} \quad \text{(in spherical coordinates again)}$$

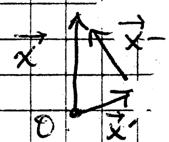
proof: pp 86b, 86d

$$\vec{E} = -\vec{\nabla}\Phi(\vec{x}) = -\left[\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right]$$

Jackson: "exercise fi... the reader"

Proof:

$$4\pi\varepsilon_0 \Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$



Taylor expand $\frac{1}{|\vec{x}-\vec{x}'|}$. Expand in $\underbrace{(x', y', z')}_{\vec{x}'}$ about $\underbrace{(x, y, z)}_{\vec{x}}$ (fixed point).

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} + \left[\frac{\partial}{\partial x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(x') + \left[\frac{\partial}{\partial y'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(y') +$$

$$\left[\frac{\partial}{\partial z'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(z') + \frac{1}{2!}\left[\frac{\partial^2}{\partial x'^2}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(x')^2 +$$

$$\frac{1}{2!}\left[\frac{\partial^2}{\partial x'\partial y'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(x'y') + \frac{1}{2!}\left[\frac{\partial^2}{\partial x'\partial z'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]\Bigg|_{\vec{x}'=0}(x'z') + \ldots$$

$$\frac{\partial}{\partial x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\Bigg|_{\vec{x}'=0} = -\frac{1}{\cancel{2}}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}\cdot\cancel{2}(x-x')(-1)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}\cdot x$$

$$\frac{\partial^2}{\partial x'^2}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\partial}{\partial x'}\left[\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}(x-x')\right]$$

$$= -\frac{3}{\cancel{2}}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-5/2}\cdot\cancel{2}(x-x')(-1)(x-x')$$

$$+\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}(-1)$$

$$\underline{\vec{x}'=0:} \Rightarrow = \frac{3}{(x^2+y^2+z^2)^{5/2}}\cdot x^2 + -\frac{1}{(x^2+y^2+z^2)^{3/2}} = \frac{3x^2-(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2}{\partial x'\partial y'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\partial}{\partial y'}\left[\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}(x-x')\right]$$

$$= -\frac{3}{\cancel{2}}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-5/2}(x-x')\cancel{2}(y-y')(-1)$$

$$\underline{\vec{x}'=0:} \Rightarrow 3\frac{1}{(x^2+y^2+z^2)^{5/2}}\cdot xy$$

So we find that:

$$4\pi\epsilon_0 \Phi(\vec{x}) = \int d^3x' \rho(\vec{x}') \left[\frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{xx'+yy'+zz'}{(x^2+y^2+z^2)^{3/2}} + \right.$$

$$+ \frac{1}{2!} \frac{[3x^2-(x^2+y^2+z^2)]\cdot(x')^2 + [3xy](x'y') + [3xz](x'z') + \ldots}{(x^2+y^2+z^2)^{5/2}}$$

Zeroth-Order Term:

$$\int d^3x' \rho(x') \frac{1}{r} = \frac{1}{r}\int \rho(x')\, d^3x' = \frac{q}{r}$$

First-Order Terms:

$$\frac{1}{r^3} \int d^3x' \rho(\vec{x}') [xx'+yy'+zz'] = \frac{x}{r^3}\int d^3x'\, x' \rho(\vec{x}') + \ldots$$

$$= \frac{x}{r^3} p_x + \frac{y}{r^3} p_y + \frac{z}{r^3} p_z = \vec{x}\cdot\vec{p}/r^3$$

Second-Order Terms:

$$\frac{1}{2!}\frac{1}{r^5} \int d^3x' \rho(\vec{x}') \Big[(3x'y')xy + (3x'z')xz + (3y'x')yx + (3y'z')yz$$

$$+ (3z'x')zx + (3z'y')zy$$

$$+ (x')^2(2x^2-y^2-z^2) + (y')^2(2y^2-x^2-z^2)$$

$$+ (z')^2(2z^2-x^2-y^2) \Big]$$

$$= \frac{1}{2!}\frac{1}{r^5} \int d^3x' \rho(\vec{x}') \Big[2\cdot(3x'y')xy + 2(3x'z')xz + 2(3y'z')yz$$

$$+ x^2(3(x')^2 - (x'^2+y'^2+z'^2)) + y^2(3(y')^2 - (x'^2+y'^2+z'^2))$$

$$+ z^2(3(z')^2 - (x'^2+y'^2+z'^2)) \Big]$$

$$= \frac{1}{2!}\frac{1}{r^5} \int d^3x' \rho(\vec{x}') \left[\sum_{i,j} (3x_i'x_j' - (r')^2 \delta_{ij}) \right] \cdot \overbrace{X_i X_j}^{\text{not primed!}}$$

$$= \frac{1}{2!}\frac{1}{r^5} \sum_{i,j} X_i X_j Q_{ij}$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j}\frac{x_i x_j}{r^5}Q_{ij} + \dots\right]$$

$r \equiv |\vec{x}|$

QED.

Note: this derivation is based on the assumption that the charge is contained within a sphere centered on the origin

If not, just translate $\vec{x} \rightarrow \vec{x} - \vec{x}'$

$$\Rightarrow E_r = \sum_{l,m} \frac{1}{\varepsilon_0} \frac{q_{lm} Y_{lm}(\theta,\phi)}{(2l+1)\, r^{l+2}} (l+1)$$

$$E_\theta = -\sum_{l,m} \frac{1}{\varepsilon_0} \frac{1}{r^{l+2}} \frac{q_{lm}}{(2l+1)} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

$$E_\phi = -\sum_{l,m} \frac{1}{\varepsilon_0} \frac{1}{r^{l+2}} \frac{q_{lm}}{(2l+1)} \frac{1}{\sin\theta} \underbrace{\frac{\partial}{\partial\phi} Y_{lm}(\theta,\phi)}_{= i m Y_{lm}(\theta,\phi) \text{ as } Y_{lm} \propto P_l^m(\cos\theta) e^{im\phi}}$$

Note: For monople, $l=0$:

$$E_r = \frac{1}{\varepsilon_0} \cdot \frac{q_{00}}{1} \frac{1}{r^2} Y_{00} \cdot (1) = \frac{1}{\varepsilon_0} \cdot \frac{q}{\sqrt{4\pi}} \frac{1}{r^2} \cdot \frac{1}{\sqrt{4\pi}} = \frac{q}{4\pi\varepsilon_0 r^2} \checkmark$$

$$E_\theta \propto \frac{\partial}{\partial\theta} Y_{00} = 0 \checkmark \; ; \quad E_\phi \propto \frac{\partial}{\partial\phi} Y_{00} = 0 \checkmark$$

For dipole (moment), $l = 1$ ($m=0$): along the $+z$-axis, i.e., $\vec{p} = (0,0,p)$, $\rightarrow$ expect azimuthal symmetry, $\underline{\underline{m=0}}$ (and p_z only appears in q_{10}!)

$$E_r = \frac{1}{\varepsilon_0} \frac{q_{10}}{3} \frac{Y_{10}(\theta,\phi)}{r^3} (2)$$

$$= \frac{2}{\cancel{3}\varepsilon_0} \cdot \frac{1}{r^3} \cdot \frac{\cancel{\sqrt{3}}}{\sqrt{4\pi}} \cdot p \cdot \sqrt{\frac{\cancel{3}}{4\pi}} \cos\theta$$

$$= \frac{1}{2\pi\varepsilon_0} \frac{p\cos\theta}{r^3} \checkmark \quad \text{(usual form)}$$

$$E_\theta = \frac{-1}{\varepsilon_0} \cdot \frac{1}{r^3} \cdot \frac{\cancel{\sqrt{3}}}{\sqrt{4\pi}} \cdot p \cdot \frac{1}{\cancel{3}} \sqrt{\frac{\cancel{3}}{4\pi}} (-\sin\theta) = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \checkmark$$

$$E_\phi \propto \frac{\partial}{\partial\phi} Y_{10}(\theta,\phi) = 0 \quad \text{(as no } \phi\text{-dependence for } m=0!) \checkmark$$

$\longleftarrow$ (See 87b)

Note: can write: $\Phi(\vec{x}) = \frac{1}{\varepsilon_0} \sum_{l,m} \frac{1}{2l+1} \left[q_{lm} \right] \cdot \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$

if charge density $\rho(\vec{x}')$ all within some sphere, or if charge density falls off faster than any power of $\frac{1}{r^n}$ (e.g., exponential)

Otherwise, need to use Green function $G(\vec{x},\vec{x}')$ or other approach. (e.g., Jackson, 4.7, $\rho(r) \propto r^2 e^{-r}$)