$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \qquad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \qquad E_\phi = 0 \quad , \quad \vec{E} = E_r\hat{r} + E_\theta\hat{\theta}$$

at origin

• This is for dipole $\vec{p}$ along $\hat{z}$. In general, could rotate coordinate system so $\vec{p}$ along $\hat{z}$! So sufficiently general! , $\vec{p} = p\hat{z}$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \end{cases}$$

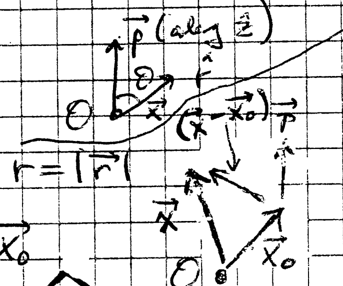
substituting: (to find more general expression)

$$\Rightarrow \vec{E}(\vec{x}) = \hat{x}\left[\frac{3p\cos\theta\sin\theta\cos\phi}{4\pi\varepsilon_0 r^3}\right] + \hat{y}\left[\frac{3p\cos\theta\sin\theta\sin\phi}{4\pi\varepsilon_0 r^3}\right] + \hat{z}\left[\frac{\overbrace{2p\cos^2\theta - p\sin^2\theta}^{=p(3\cos^2\theta - 1)}}{4\pi\varepsilon_0 r^3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[(\hat{x}\sin\theta\cos\phi)3p\cos\theta + (\hat{y}\sin\theta\sin\phi)3p\cos\theta + (\hat{z}\cos\theta)3p\cos\theta - p\hat{z}\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[3(p\cos\theta)\hat{r} - p\hat{z}\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] \quad , \quad r = |\vec{r}|$$



If dipole $\vec{p}$ not at origin, but at $\vec{x}_0$

$$\vec{r} = \vec{x} \Rightarrow (\vec{x} - \vec{x}_0) \qquad \hat{n} \equiv \widehat{(\vec{x} - \vec{x}_0)}$$

$$\Rightarrow \vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0|\vec{x} - \vec{x}_0|^3}\left[3(\vec{p}\cdot\hat{n})\hat{n} - \vec{p}\right] \checkmark$$

In general, for dipole $\vec{p}$ at point $\vec{x}_0$, the field at $\vec{x}$ is:

$$\vec{E}(\vec{x}) = \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{4\pi\varepsilon_0|\vec{x} - \vec{x}_0|^3}$$

In general, q_{lm} depend on O (i.e., choice of origin of coord. system).

General Th'm (Jackson Problem 4.4): q_{lm} for lowest non-vanishing moment (l) independent of O. But all higher ~~moments are dependent. True for all $\rho(\vec{x}')$!~~

Example of Such: Two Point Charges

$+q$ • $\vec{x}_+ \equiv (0, 0, d)$

origin O

$-q$ • $\vec{x}_- \equiv (0, 0, -d)$

$$\rho(\vec{x}') = +q\,\delta(\vec{x}' - \vec{x}_+) - q\,\delta(\vec{x}' - \vec{x}_-)$$

Monopole: $l = 0$

multipole moment

$$q_{00} = \int d^3x'\,\rho(\vec{x}')\,(r')^0\,Y^*_{00}(\theta', \phi') = \int d^3x' \left[+q\,\delta(\vec{x}' - \vec{x}_+) - q\,\delta(\vec{x}' - \vec{x}_-)\right]\frac{1}{\sqrt{4\pi}}$$

$$= \frac{1}{\sqrt{4\pi}}(+q - q) = \underline{\underline{0}} \checkmark \quad \text{(just total charge)}$$

Dipole Moment $\vec{p}$ (vector):

$$\vec{p} = \int \vec{x}'\,\rho(\vec{x}')\,d^3x' = \left[\vec{x}_+ q - \vec{x}_- q\right]$$

$$\vec{x}_- = -\vec{x}_+ \Rightarrow \underline{\underline{\vec{p} = 2q\vec{x}_+}} \checkmark = 2qd\hat{z} \qquad \vec{x}_+ = d\hat{z} \quad \text{so only } q_{10} \neq 0$$

lowest non-vanishing q_{lm}

Quadrupole:

$$Q_{ij} \equiv \int d^3x'\,\rho(\vec{x}')\left[3x_i'x_j' - (r')^2\delta_{ij}\right] \quad \text{(tensor term)}$$

e.g.:

$$Q_{xx} = \int d^3x' \left[+q\,\delta(\vec{x}' - \vec{x}_+) + -q\,\delta(\vec{x}' - \vec{x}_-)\right] \cdot \left[3x'^2 - (r')^2\right]$$

$$= \left[+q(3 \cdot 0^2) - q(3 \cdot 0^2)\right] + \left[+q(-d^2) - q(-d^2)\right] = 0 \;//$$

But! If charges at:

$+q$ • $(0, 0, 2d)$

$-q$ • $(0, 0, 0)$ ← origin

$$\Rightarrow \text{then } Q_{xx} = (4q \cdot 0 - q \cdot 0) + \left[q(-4d^2) - q(0)\right]$$

$$\neq 0! \;//$$

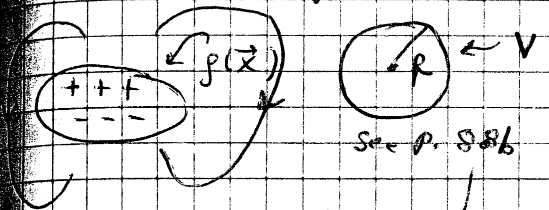
but $\vec{p} = 2qd\hat{z}$ still invariant!

See Jackson pp. 147–148

Consider topic which will arise again when study magnetic dipoles.

Consider charge distribution, $\rho(\vec{x})$, $\Rightarrow$ $\vec{E}(\vec{x})$ throughout space.

Calculate: integral of $\vec{E}$ over volume of sphere of radius R.



See p. 88b

Can show: $\displaystyle\int_{\substack{\text{sphere}\\ \text{radius } R,\ (V)}} \vec{E}(\vec{x})\, d^3x = -\oint_{\substack{\text{Surface}\\ \text{sphere},\ (S)}} R^2 d\Omega\, \Phi(\vec{x})\, \hat{n}$, $\left[\hat{n} = \dfrac{\vec{r}}{|\vec{r}|} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{R} = \vec{x}/R \right]$ on sphere surface

Recall: $\displaystyle \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|}$ } recall: integral over all space

$$\Rightarrow \int_V \vec{E}(\vec{x})\, d^3x = -\oint_S R^2 d\Omega\, \Phi(\vec{x})\, \hat{n}$$

$$= \frac{-R^2}{4\pi\varepsilon_0} \int d^3x'\, \rho(\vec{x}') \oint_S d\Omega \frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$ decompose in terms of Y_{lm}'s

Recall in spherical coordinates: $\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$
(obvious, for sphere of radius 1)

But! Note: $Y^*_{10}(\theta,\phi) = \dfrac{\sqrt{3}}{\sqrt{4\pi}} \cos\theta$

$$Y^*_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

$$\left(Y_{1,-1}(\theta,\phi)\right)^* = \left[(-1)^1 Y^*_{11}(\theta,\phi)\right]^* = +\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{+i\phi}$$

$$\Rightarrow \hat{n} = \sqrt{\frac{8\pi}{3}} \left[\frac{-Y^*_{11} + Y^*_{1,-1}}{2}\right]\hat{x} + \sqrt{\frac{8\pi}{3}}\left[\frac{+Y^*_{1,1} + Y^*_{1,-1}}{+2i}\right]\hat{y} + \sqrt{\frac{4\pi}{3}}\, Y^*_{10}\,\hat{z}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{1}{(2l+1)} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

$\vec{x}$: on S (surface sphere)
$\vec{x}'$: charges

$r_<$ ($r_>$) smaller (greater) of $|\vec{x}|$ and $|\vec{x}'|$

$$\left[\begin{array}{l} V = \text{sphere} \\ S = \text{sphere surface} \end{array}\right.$$

$$\int_V \vec{E}(\vec{x})\, d^3x = -\int_V \vec{\nabla}\Phi(\vec{x})\, d^3x$$

$$= -\left[\int_V \left(\frac{\partial \Phi}{\partial x}\hat{x} + \frac{\partial \Phi}{\partial y}\hat{y} + \frac{\partial \Phi}{\partial z}\hat{z}\right) d^3x\right]$$

Consider: $\displaystyle\int_V \frac{\partial \Phi}{\partial x}\, d^3x = \int_V (\vec{\nabla}\Phi \cdot \hat{x})\, d^3x$

[Recall: Integration by parts in > 1 dimension (p. 28):

$$\int_V (\vec{\nabla} u)\cdot \vec{v}\, d^3x = \oint_S (u\vec{v})\cdot \hat{n}\, da - \int_V u(\vec{\nabla}\cdot\vec{v})\, d^3x$$

In above, $u = \Phi$, $V = \hat{x}$, $\hat{n} = \vec{x}/R \longrightarrow$ (p. 88)

$$\Rightarrow \int_V (\vec{\nabla}\Phi\cdot\hat{x})\, d^3x = \oint_S \Phi\,\hat{x}\cdot\frac{\vec{x}}{R}\, R^2\, d\Omega - \int_V \Phi\,\underbrace{(\vec{\nabla}\cdot\hat{x})}_{=0}\, d^3x$$

$$= \oint_S \Phi(\vec{x})\,\hat{x}\cdot\vec{x}\; R\, d\Omega \qquad \left[\vec{\nabla}\cdot 1\hat{x} = 0\right]$$

Similarly, $\displaystyle\int_V (\vec{\nabla}\Phi\cdot\hat{y})\, d^3x = \oint_S \Phi(\vec{x})\,\hat{y}\cdot\vec{x}\; R\, d\Omega$

$$\int_V (\vec{\nabla}\Phi\cdot\hat{z})\, d^3x = \oint_S \Phi(\vec{x})\,\hat{z}\cdot\vec{x}\; R\, d\Omega$$

$$\Rightarrow -\int \vec{\nabla}\Phi(\vec{x})\, d^3x = -\oint_S \left[\Phi(\vec{x})\hat{x} + \Phi(\vec{x})\hat{y} + \Phi(\vec{x})\hat{z}\right]\cdot\vec{x}\, R\, d\Omega$$

$$= -\oint_S \Phi(\vec{x})\,\underbrace{\left[x\hat{x} + y\hat{y} + z\hat{z}\right]}_{=\hat{n}R}\, R\, d\Omega$$

$$= -\oint_S \Phi(\vec{x})\,\hat{n}\, R^2\, d\Omega \quad ✓$$