$$\Rightarrow -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\, \rho(\vec{x}')\oint_S d\Omega\,\frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$

$$-\frac{R^2}{4\pi\epsilon_0}\int d^3x'\,\rho(\vec{x}')\cdot\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\cdot\overbrace{\hat{n}\left(Y_{10}^*(\theta,\phi),\,Y_{11}^*(\theta,\phi),\,Y_{1,-1}^*(\theta,\phi)\right)}^{\text{"function of"}}$$

$$\cdot\, 4\pi\sum_{l,m}\frac{1}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}\,\underbrace{Y_{lm}^*(\theta',\phi')}_{\text{not integration variables}}Y_{lm}(\theta,\phi)$$

Recalling: $\displaystyle\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\, Y_{l'm'}^*(\theta,\phi)\,Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$

$\Rightarrow$ in $\sum_{l,m}$ only $(l,m) = (1,0),\ (1,1)$, and $(1,-1)$ terms survive!

$$\Rightarrow\oint_S\frac{d\Omega\,\hat{n}}{|\vec{x}-\vec{x}'|} = \int d\Omega\sqrt{\frac{8\pi}{3}}\,\hat{x}\left[\frac{-Y_{11}^*(\theta,\phi)+Y_{1,-1}^*(\theta,\phi)}{2}\right]\cdot 4\pi\,\frac{1}{3}\,\frac{r_<^l}{r_>^2}\Big(Y_{11}^*(\theta',\phi')Y_{11}(\theta,\phi) + Y_{1,-1}^*(\theta',\phi')Y_{1,-1}(\theta,\phi)\Big)$$

$$+\int d\Omega\sqrt{\frac{8\pi}{3}}\,\hat{y}\left[\frac{+Y_{11}^*(\theta,\phi)+Y_{1,-1}^*(\theta,\phi)}{2i}\right]\cdot 4\pi\,\frac{1}{3}\,\frac{r_<^1}{r_>^2}\Big(Y_{11}^*(\theta',\phi')Y_{11}(\theta,\phi) + Y_{1,-1}^*(\theta',\phi')Y_{1,-1}(\theta,\phi)\Big)$$

$$+\int d\Omega\sqrt{\frac{4\pi}{3}}\,\hat{z}\left[Y_{10}^*(\theta,\phi)\right]\cdot 4\pi\,\frac{1}{3}\cdot\frac{r_<^1}{r_>^2}\,Y_{10}^*(\theta',\phi')Y_{10}(\theta,\phi)$$

$$= \hat{x}\sqrt{\frac{8\pi}{3}}\cdot\frac{1}{2}\cdot\frac{4\pi}{3}\cdot\frac{r_<^1}{r_>^2}\left(-Y_{11}^*(\theta',\phi')+Y_{1,-1}^*(\theta',\phi')\right)$$

$$+\hat{y}\sqrt{\frac{8\pi}{3}}\,\frac{1}{2i}\,\frac{4\pi}{3}\,\frac{r_<^1}{r_>^2}\left(+Y_{11}^*(\theta',\phi')+Y_{1,-1}^*(\theta',\phi')\right)$$

$$+\hat{z}\sqrt{\frac{4\pi}{3}}\,\frac{4\pi}{3}\,\frac{r_<^1}{r_>^2}\,Y_{10}^*(\theta',\phi')$$

$$\left[\begin{aligned} +Y_{11}^*(\theta',\phi') &= -\sqrt{\frac{3}{8\pi}}\sin\theta' e^{-i\phi'} \\ Y_{1,-1}^*(\theta',\phi') &= \sqrt{\frac{3}{8\pi}}\sin\theta' e^{+i\phi'}\end{aligned}\qquad Y_{10}^*(\theta',\phi') = \sqrt{\frac{3}{4\pi}}\cos\theta'\right]$$

$$= \hat{x}\left[\frac{4\pi}{3}\sin\theta'\cos\phi'\cdot\frac{r_<^1}{r_>^2}\right]+\hat{y}\left[\frac{4\pi}{3}\sin\theta'\sin\phi'\,\frac{r_<^1}{r_>^2}\right]+\hat{z}\left[\frac{4\pi}{3}\cos\theta'\,\frac{r_<^1}{r_>^2}\right]$$

$$= \frac{4\pi}{3}\cdot\frac{r_<^1}{r_>^2}\left[\sin\theta'\cos\phi'\,\hat{x}+\sin\theta'\sin\phi'\,\hat{y}+\cos\theta'\,\hat{z}\right] = \frac{4\pi}{3}\,\frac{r_<^1}{r_>^2}\,\hat{n}'\ ;\quad \hat{n}' \equiv \frac{\vec{r}'}{|\vec{r}'|}$$

$$\Rightarrow \int d\Omega \, \frac{\hat{n}}{|\vec{x}-\vec{x}'|} = \frac{4\pi}{3}\,\frac{r_<}{r_>^2}\,\hat{n}'$$

$$\int_V \vec{E}(\vec{x})\, d^3x = -\frac{R^2}{4\pi\epsilon_0}\int d^3x'\,\rho(\vec{x}') \oint_S d\Omega\,\frac{\hat{n}}{|\vec{x}-\vec{x}'|}$$

$$= -\frac{R^2}{\cancel{4\pi}\epsilon_0}\int d^3x'\,\rho(\vec{x}')\,\frac{\cancel{4\pi}}{3}\,\frac{r_<}{r_>^2}\,\hat{n}'$$

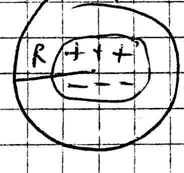
$$= -\frac{R^2}{3\epsilon_0}\int d^3x'\,\rho(\vec{x}')\,\frac{r_<}{r_>^2}\,\hat{n}'$$

sphere radius

$r_<$ ($r_>$) smaller (greater) of R and r'

charges

• If sphere of radius R completely encloses the charges,



$r_< = r'$, $r_> = R$

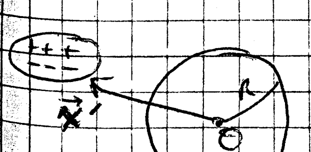
$$\Rightarrow \int_V \vec{E}(\vec{x})\, d^3x = \frac{-\cancel{R^2}}{3\epsilon_0}\int d^3x'\,\rho(\vec{x}')\,\frac{r'}{\cancel{R^2}}\,\hat{n}'$$

$$= -\frac{1}{3\epsilon_0}\int d^3x'\,\rho(\vec{x}')\, r'\,\hat{n}'$$

$$= -\frac{1}{3\epsilon_0}\underbrace{\int d^3x'\,\rho(\vec{x}')\,\vec{r}'}_{=\vec{p}}$$ (electric dipole moment)

$$\Rightarrow \int_V \vec{E}(\vec{x})\, d^3x = -\frac{\vec{p}}{3\epsilon_0} \quad (*)$$

• If charge external to sphere, $r_> = r'$, $r_< = R$



$$\int_V \vec{E}(\vec{x})\, d^3x = \frac{-R^2}{3\epsilon_0}\int d^3x'\,\rho(\vec{x}')\,\frac{R}{(r')^2}\,\hat{n}'$$

$$= -\frac{R^3}{3\epsilon_0}\underbrace{\int d^3x'\,\frac{\rho(\vec{x}')}{(r')^2}\,\hat{n}'}, \quad \hat{n}' = \frac{\vec{x}'}{|\vec{x}'|}$$

= integral over charge distribution, $\rho(\vec{x}')$

Recall: $\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, d^3x'}$

$$= -4\pi\epsilon_0\,\vec{E}(0)\ !!$$

$$\Rightarrow \vec{E}(\vec{0}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\frac{(-\vec{x}')}{|\vec{x}'|^3}\, d^3x'$$

$\vec{x} = \vec{0}$

$$\Rightarrow \int_V \vec{E}(\vec{x})\, d^3x = \frac{4\pi}{3} R^3\, \vec{E}(0) \quad (**) \quad \Rightarrow \quad \frac{\int \vec{E}(\vec{x})\, d^3x}{\frac{4}{3}\pi R^3} = \vec{E}(0)$$ (91)

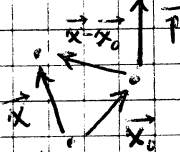
$\Rightarrow$ Proves average value of $\vec{E}$ over a spherical volume containing no charge is the value of the field at the center of the sphere. ✓

[Con]cerning $(*)$, previous page, $\int_V \vec{E}(\vec{x})\, d^3x = -\frac{\vec{p}}{3\epsilon_0}$ for charge inside sphere

$\vec{p}$: EDM of $\rho(\vec{x}')$ w.r.t. center of sphere!

But we had from before:

let: $\vec{E}_1(\vec{x}) \equiv \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{4\pi\epsilon_0 |\vec{x}-\vec{x}_0|^3}$ $\begin{cases} \vec{p} : \text{dipole at } \vec{x}_0 \\ \hat{n} : \text{unit vector from } \vec{x}_0 \text{ to } \vec{x} \end{cases}$



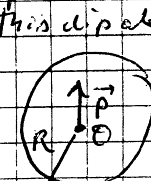
and if sphere contains $\vec{p}$, $\int \vec{E}_1(\vec{x})\, d^3x = 0$

e.g., for dipole at $(0,0,0)$ of $\vec{p} = (0,0,p)$, sphere radius R surrounding this dipole

$E_r = \frac{1}{2\pi\epsilon_0} \frac{p\cos\theta}{r^3}$

$E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$

$E_\phi = 0$



$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta}$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

$$\Rightarrow \vec{E}_1 = \frac{1}{2\pi\epsilon_0 r^3} p\cos\theta\sin\theta\cos\phi\,\hat{x} + \frac{1}{2\pi\epsilon_0 r^3} p\cos\theta\sin\theta\sin\phi\,\hat{y} + \frac{1}{2\pi\epsilon_0 r^3} p\cos^2\theta\,\hat{z}$$

$$+ \frac{p}{4\pi\epsilon_0 r^3}\sin\theta\cos\theta\cos\phi\,\hat{x} + \frac{p}{4\pi\epsilon_0 r^3}\cos\theta\sin\theta\sin\phi\,\hat{y} + \frac{-p}{4\pi\epsilon_0 r^3}\sin^2\theta\,\hat{z}$$

$$\text{In } \int_V \vec{E}_1(\vec{x})\, d^3x = \underbrace{\int_0^{2\pi} d\phi(\ldots)}_{\hat{x} \text{ and } \hat{y} \text{ terms } \to 0} \underbrace{\int_0^{\pi} \sin\theta\, d\theta(\ldots)}_{\hat{z} : (2\cos^2\theta - \sin^2\theta)\sin\theta} \cdot \underbrace{\int_0^R r^2 dr(\ldots)}_{\propto r^2 \frac{1}{r^3} dr}$$

$= (3\cos^2\theta - 1)\sin\theta$

$\int \Rightarrow -\cos^3\theta + \cos\theta = 0$

integrate to 0

$\propto \ln r$

bad at $r=0$

Convention:

$$\int_V \vec{E}_1(\vec{x})\, d^3x = 0$$ (singularity at $\vec{x} = \vec{x}_0$ leads to ambiguous result).

Then to satisfy (*) that

$$\int_V \vec{E}(\vec{x})\, d^3x = -\frac{\vec{p}}{3\epsilon_0},$$ define:

$\vec{E}$-field of dipole is:

$$\vec{E}(\vec{x}) \equiv \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]$$

$\Rightarrow$ if $\vec{p}$ inside sphere, i.e., $\vec{x}_0$ inside sphere

$\int d^3x$ (first term) $= 0$ by convention,

$\int d^3x$ (second term) $= -\frac{\vec{p}}{3\epsilon_0}$ ✓

Important result: dipole $\overset{\text{moment}}{\vec{p}}$, arising from $\overset{\text{extended}}{\text{charge}}$ distribution $\rho(\vec{x}')$, where

$$\vec{p} = \int d^3x'\, \rho(\vec{x}')\,\vec{x}'$$

can be treated as if point dipoles, with δ-function term carrying information about actual finite distribution of charge!

$\Rightarrow$ Will also come up when treating magnetic dipoles!!

Multipole Expansion of Interaction Energy of Charge Distribution in External Field

Recall: If some $\rho(\vec{x})$ placed in an externally-generated $\Phi(\vec{x})$

$$W = \int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

interaction electrostatic energy

[continuous version of discrete $W = \sum_i q_i \Phi_i$]

If $\Phi(\vec{x})$ varies "slowly" over region where $\rho(\vec{x}) \neq 0$:

Taylor Expand: $$\Phi(\vec{x}) = \Phi(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\Phi\Big|_{\vec{x}=0} + \frac{1}{2!}\left[\sum_{i,j} x_i x_j \frac{\partial^2\Phi}{\partial x_i \partial x_j}\Big|_{\vec{x}=0}\right] + \dots$$