As $\vec{E} = -\vec{\nabla}\Phi = -\frac{\partial\Phi}{\partial x}\hat{x} + -\frac{\partial\Phi}{\partial y}\hat{y} - \frac{\partial\Phi}{\partial z}\hat{z}$

$$\Phi(\vec{x}) = \Phi(\vec{x}=0) - \vec{x}\cdot\vec{E}(0) + \frac{1}{2}\left[\sum_{i,j} x_i x_j \frac{\partial}{\partial x_i}\Big(\overbrace{\frac{\partial\Phi}{\partial x_j}}^{=-E_j(0)}\Big)\Big|_{\vec{x}=0}\right]$$

$$= \Phi(\vec{x}=0) - \vec{x}\cdot\vec{E}(\vec{x}=0) - \frac{1}{2}\sum_{i,j} x_i x_j \frac{\partial E_j(\vec{x}=0)}{\partial x_i}$$

$\vec{\nabla}\cdot\vec{E} = 0$ for the external field in which our charge is placed (interacting with)

$$\sum_{i,j} x_i x_j \frac{\partial E_j(0)}{\partial x_i} = x^2 \frac{\partial E_x}{\partial x}\Big|_0 + xy\frac{\partial E_x}{\partial y}\Big|_0 + xz\frac{\partial E_x}{\partial z}\Big|_0$$
$$+ y^2\frac{\partial E_y}{\partial y}\Big|_0 + yx\frac{\partial E_y}{\partial x}\Big|_0 + yz\frac{\partial E_y}{\partial z}\Big|_0$$
$$+ z^2\frac{\partial E_z}{\partial z}\Big|_0 + zx\frac{\partial E_z}{\partial x}\Big|_0 + zy\frac{\partial E_z}{\partial y}\Big|_0$$

$$r^2\,\vec{\nabla}\cdot\vec{E}(0) = (x^2+y^2+z^2)\left(\frac{\partial E_x}{\partial x}\Big|_0 + \frac{\partial E_y}{\partial y}\Big|_0 + \frac{\partial E_z}{\partial z}\Big|_0\right) = 0$$
$$= x^2\frac{\partial E_x}{\partial x}\Big|_0 + x^2\frac{\partial E_y}{\partial y}\Big|_0 + x^2\frac{\partial E_z}{\partial z}\Big|_0$$
$$+ y^2\frac{\partial E_x}{\partial x}\Big|_0 + y^2\frac{\partial E_y}{\partial y}\Big|_0 + z^2\frac{\partial E_z}{\partial z}\Big|_0$$
$$+ z^2\frac{\partial E_x}{\partial x}\Big|_0 + z^2\frac{\partial E_y}{\partial y}\Big|_0 + z^2\frac{\partial E_z}{\partial z}\Big|_0$$

$$\Rightarrow \sum_{i,j} x_i x_j \frac{\partial E_j(0)}{\partial x_i} - \frac{1}{3}r^2\,\overbrace{\vec{\nabla}\cdot\vec{E}(0)}^{=0}$$
$$= x^2\frac{\partial E_x}{\partial x}\Big|_0 - \frac{1}{3}(x^2+y^2+z^2)\frac{\partial E_x}{\partial x}\Big|_0$$
$$+ y^2\frac{\partial E_y}{\partial y}\Big|_0 - \frac{1}{3}(x^2+y^2+z^2)\frac{\partial E_y}{\partial y}\Big|_0$$
$$+ z^2\frac{\partial E_z}{\partial z}\Big|_0 - \frac{1}{3}(x^2+y^2+z^2)\frac{\partial E_z}{\partial z}\Big|_0 + \sum_{i\neq j} x_i x_j \frac{\partial E_j(0)}{\partial x_i}$$
$$= \frac{1}{3}\sum_{i,j}\left[3x_i x_j \frac{\partial E_j}{\partial x_i}\Big|_0 - r^2\frac{\partial E_j(0)}{\partial x_i}\delta_{ij}\right]$$

$$\Rightarrow \Phi(\vec{x}) = \Phi(\vec{x}=0) - \vec{x}\cdot\vec{E}(\vec{x}=0) - \frac{1}{2}\cdot\frac{1}{3}\sum_{i,j}\left[3x_i x_j\frac{\partial E_j(0)}{\partial x_i} - r^2\frac{\partial E_j(0)}{\partial x_i}\delta_{ij}\right]$$
$$= \Phi(\vec{x}=0) - \vec{x}\cdot\vec{E}(\vec{x}=0) - \frac{1}{6}\sum_{i,j}\left[3x_i x_j - r^2\delta_{ij}\right]\frac{\partial E_j(0)}{\partial x_i}$$

Now calculate interaction energy W!

$$W = \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

$$= \int \rho(\vec{x})\, d^3x \left[\Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_j(0)}{\partial x_i} + \dots \right]$$

$$= \int d^3x\, \rho(\vec{x})\, \Phi(0) - \underbrace{\int d^3x\, \rho(\vec{x})\, \vec{x}}_{=\vec{p}} \cdot \vec{E}(0) - \frac{1}{6}\int d^3x\, \rho(\vec{x}) \left[\sum_{i,j}\left(3x_i x_j - r^2\delta_{ij}\right) \cdot \frac{\partial E_j(0)}{\partial x_i} \right]$$

$$= q\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + \dots$$

Physical Interpretation:
- charge interacts with potential
- dipole interacts with field
- quadrupole interacts with field gradient
- ⋮ etc.

Electrostatics in Media

So far, considered $\Phi(\vec{x})$ and $\vec{E}$ in presence of charges and conductors, but no other media. (just vacuum)

In media, responses of charges and fields must be accounted for.

Later on, average carefully over microscopically large, but macroscopically small, regions to obtain the macroscopic Maxwell equations.

Elementary discussion of macroscopic quantities: $\vec{E}$, Φ

- $\vec{\nabla}\times\vec{E}_{micro} = 0$, still have $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E}$ still derivable from $\Phi(\vec{x})$ in macroscopic case

- ~~If~~ Apply $\vec{E}$-field to media of atoms or molecules, molecular/atomic charge density will be distorted!
  Simple substances, multipole moments $= 0$ averaged over many molecules, for $\vec{E} = 0$,
  → Dominant multipole in presence of $\vec{E}$-field will be dipole.

  $\Longrightarrow$ produces "electric polarization" (dipole moment/unit volume): $\vec{P}(\vec{x}) = \sum_i N_i \langle \vec{p}_i \rangle$ — $\vec{p}_i$: dipole moment of i^{th} molecule; N_i: number density

  $\Longrightarrow$ if molecules have net charge e_i, and if there is macroscopic excess or free charge,

  macroscopic $\rho(\vec{x}) = \sum_i N_i \underbrace{\langle e_i \rangle}_{\text{usually } = 0 \text{ for atoms/molecules} \dots} + \rho_{excess}$

Build up macroscopic potential and fields:

- charge of some $\Delta V = \rho(\vec{x}')\Delta V$ at point $\vec{x}'$
- dipole moment of some $\Delta V = \vec{P}(\vec{x}')\Delta V$ at point $\vec{x}'$

If no higher macroscopic multipoles: potential $\Delta\Phi(\vec{x},\vec{x}')$ due to moments in ΔV is:

$$\Delta\Phi(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0}\left[\frac{\rho(\vec{x}')\Delta V}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\Delta V\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] \quad \left(\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0 r} + \frac{\vec{p}\cdot\vec{x}}{4\pi\epsilon_0 r^3} + \right.$$

The $\Delta V \to d^3x'$, $\Delta\Phi(\vec{x},\vec{x}') \to d\Phi(\vec{x},\vec{x}')$, recall

$$\Phi(\vec{x},\vec{x}') = \frac{1}{4\pi\epsilon_0}\int d^3x'\left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \vec{P}(\vec{x}')\cdot\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]$$

(see p. 12 of lecture notes)

Now,

$$\int_{\text{all space}} d^3x'\, \underset{(\vec{v})}{\vec{P}(\vec{x}')}\cdot\vec{\nabla}_{x'}\underset{(u)}{\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)} = \int_{S \text{ at } \infty} da'\, \underbrace{\frac{1}{|\vec{x}-\vec{x}'|}\vec{P}(\vec{x}-\vec{x}')\cdot\hat{n}}_{\to 0 \text{ at } \infty} - \int_{\text{all space}} \frac{1}{|\vec{x}-\vec{x}'|}\left(\vec{\nabla}_{x'}\cdot\vec{P}(\vec{x}')\right)d^3x'$$

(see p. 88b)

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}_{x'}\cdot\vec{P}(\vec{x}')$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x'\frac{1}{|\vec{x}-\vec{x}'|}\left[\underbrace{\rho(\vec{x}') - \vec{\nabla}_{x'}\cdot\vec{P}(\vec{x}')}\right]$$

just usual integral expression for $\Phi(\vec{x})$ as potential due to charge distribution $\rho(\vec{x}') - \vec{\nabla}_{x'}\cdot\vec{P}(\vec{x}')$!

With the usual $\vec{\nabla}\cdot\vec{E} = (\text{charge distribution})/\epsilon_0$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left[\rho(\vec{x}) - \vec{\nabla}\cdot\vec{P}(\vec{x})\right]}$$

$$\vec{\nabla}\cdot\underbrace{\left[\epsilon_0\vec{E} + \vec{P}\right]}_{\equiv\vec{D}} = \rho$$

$$\underline{\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}} \quad \text{"electric displacement"}$$

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho}$$

(macroscopic counterparts of $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$)