Scalar Potential

Let's now consider the curl of $\vec{E}$, $\vec{\nabla}\times\vec{E}$.

Recall:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}\,')(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}\,d^3x'$$

Look at the integrand:

$$\rho(\vec{x}\,')\underbrace{\frac{(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}}$$

$$\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|^3} = -\vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right) = -\vec{\nabla}_{\vec{x}}\left(\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}}\right)$$

take $\vec{\nabla}$ w.r.t. $\vec{x}$ variables (not $\vec{x}\,'$ variables)

$\vec{x}$ is the variable, $\vec{x}\,'$ is a parameter

$$\Rightarrow \vec{E}(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}\,')\,\vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}\,'|}\right)d^3x'$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}_{\vec{x}}\underbrace{\int\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x'}$$

$$\Rightarrow \vec{E}(\vec{x}) \propto -\vec{\nabla}\,(\text{scalar function})$$

$$[\vec{\nabla}\times(\vec{\nabla}\psi)]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial\psi}{\partial x_k} = \underbrace{\epsilon_{ijk}\frac{\partial^2\psi}{\partial x_j\partial x_k}}_{\text{antisymmetric}} = 0$$

And we know from vector calculus that: $\vec{\nabla}\times(\vec{\nabla}\psi) = 0$ (for not cons of function)

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$$

Let's define: $\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$, $\quad \Phi(\vec{x}) = \underbrace{\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}\,d^3x'}_{\text{integration over all charges in the problem}} + \begin{pmatrix}\text{arbitrary}\\ \text{constant}\end{pmatrix}$

Physical interpretation:

Consider charge in an E field: $\vec{F} = q\vec{E}$

Work done in moving a test charge from point 1 to point 2 against the action of the field:

$$W = -\int_1^2 \vec{F}\cdot d\vec{l} = -q\int_1^2\vec{E}\cdot d\vec{l}$$

$$= \underbrace{-q\int_1^2(-\vec{\nabla}\Phi)\cdot d\vec{l} = q\int_1^2 d\Phi}_{\text{independent of path}} = q(\Phi_2-\Phi_1)$$

implies indep't of "path": potential difference $V = -\int_1^2\vec{E}\cdot d\vec{l}$

$\Rightarrow$ $q\Phi$ can be interpreted as the potential energy of the test charge in the electrostatic $\vec{E}$ field (Φ_1 at ∞, where $\Phi = 0$ @ ∞).
if charge brought from ∞ to point $\vec{x}$

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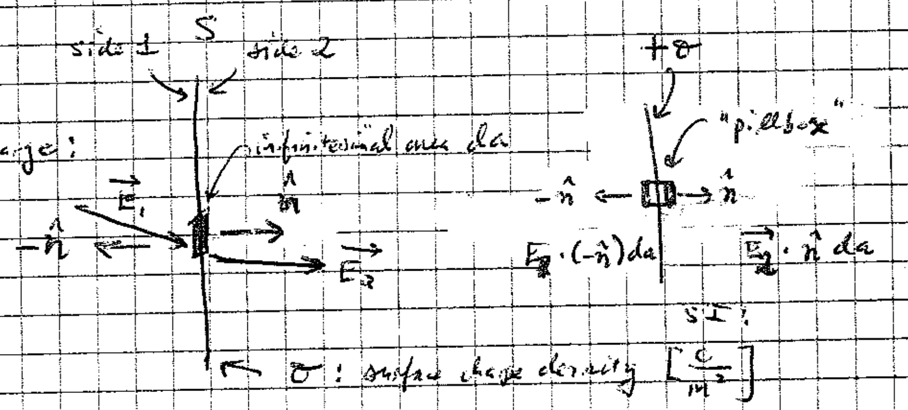
$\boxed{\text{Lecture \#3}}$

$\vec{E}$ at surface charge distribution

Let's consider a surface charge:

$\hat{n}$ defined: from 1 to 2
(e.g., $\hat{n} = \hat{x}$)



Recall Gauss's Law:

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} \underbrace{\int_V \rho(\vec{x}')\, d^3x'}_{\text{enclosed charge}} = \frac{1}{\varepsilon_0} \underbrace{\int_S \sigma(\vec{x}')\, d^2x'}_{\text{enclosed charge}}$$

~~Consider one infinitesimal area element da; (integrals over this)~~

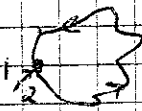
$$(\vec{E}_2 - \vec{E}_1)\cdot \hat{n}\, da = \frac{1}{\varepsilon_0} \cdot \sigma \cdot da$$

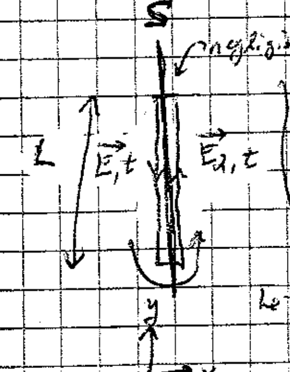
$$\Rightarrow (\vec{E}_2 - \vec{E}_1)\cdot \hat{n} = \frac{\sigma}{\varepsilon_0} \Rightarrow$$

$\boxed{\text{discontinuity in the normal component of the } \vec{E} \text{ field across a surface with a charge density (crossing in direction of } \hat{n}\text{) from 1 to 2}}$

What about the tangential component?

Recall showed: $W = -q \int_1^2 \vec{E} \cdot d\vec{l} = q(\Phi_2 - \Phi_1) \Rightarrow$ independent of path

$\Rightarrow$ if  (closed path), $-q \oint_1^2 \vec{E}\cdot d\vec{l} = q(\Phi_2 - \Phi_1) \underset{\text{if } 1 = 2}{\overset{\text{"}0\text{"}}{\Longrightarrow}} \oint_{\substack{\text{closed}\\\text{path}}} \vec{E} \cdot d\vec{l} = 0$



$$\oint \vec{E} \cdot d\vec{l} = \int_0^L E_{1,t}\, \hat{y} \cdot (-\overset{dy}{\hat{y}}) + \int_0^L E_{2,t}\, \hat{y} \cdot \overset{dy}{\hat{y}} = 0$$

per sign convention

$$-E_{1,t}L + E_{2,t}L = 0$$

$$\Rightarrow E_{1,t} = E_{2,t} \qquad \Rightarrow \vec{E}_{1,t} = \vec{E}_{2,t}$$

Let: $\vec{E}_{1,t} = E_{1,t}\,\hat{y}$
$\vec{E}_{2,t} = E_{2,t}\,\hat{y}$

$\boxed{\text{tangential component continuous across a surface charge density}}$

$$\vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \vec{\nabla}\left(\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}}\right)$$

$$\Rightarrow \quad \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$$

$$\frac{\partial\phi}{\partial x} = -\frac{1}{2}\,\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}\,2(x-x') = \frac{-(x-x')}{|\vec{x}-\vec{x}'|^3}$$

similar for $\partial_y\phi$ and $\partial_z\phi$ ....

$$\Rightarrow \quad \vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{-(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

[Note: $\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\vec{\nabla}_{\vec{x}}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$]

Taylor Expansion:

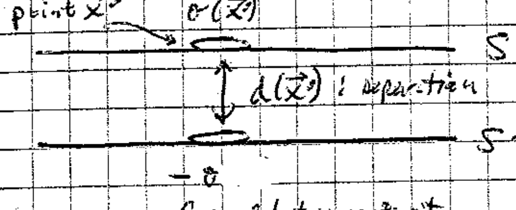
$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz$$

$$= \vec{\nabla}\Phi \cdot d\vec{l} \quad \checkmark$$

$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}, \quad \vec{\nabla}\Phi = \partial_x\Phi\,\hat{x} + \partial_y\Phi\,\hat{y} + \partial_z\Phi\,\hat{z}$$

$\Phi(\vec{x})$ at

Now consider surface dipole layer distribution:



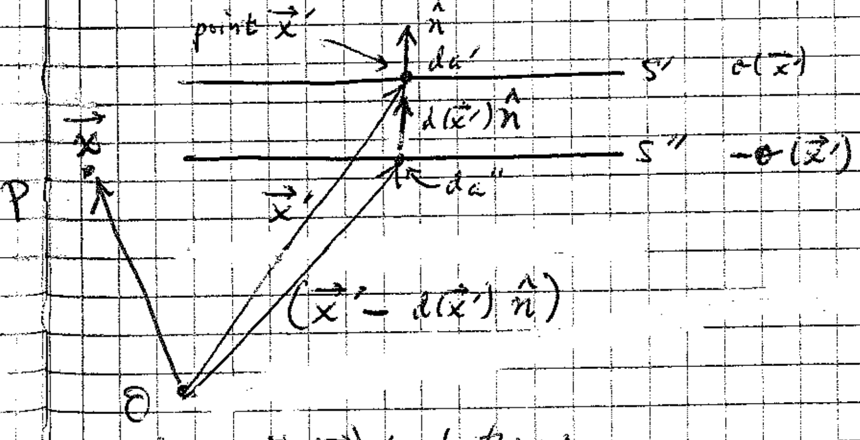
$[d(\vec{x}) \to 0]$

Dipole layer distribution of strength $D(\vec{x})$: formed by letting S' approach infinitesimally close to S, while $\sigma(\vec{x})$ becomes infinite, such that:

$$\lim_{d(\vec{x}) \to 0} \sigma(\vec{x})\, d(\vec{x}) = \underbrace{D(\vec{x})}$$

"dipole moment of the layer"
normal to S, from negative to positive charge

Find potential $\Phi(\vec{x})$ due to such a dipole layer:



Calculate $\Phi(\vec{x})$ due to this:
At some arbitrary point $\vec{x}$,

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\, da' - \frac{1}{4\pi\varepsilon_0}\int_{S''} \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'+d(\vec{x}')\hat{n}|}\, da''$$

$$|\vec{x} - (\vec{x}' - d(\vec{x}')\,\hat{n})|$$

Consider small $d = d(\vec{x}')$:

$$\frac{1}{|\vec{x}-\vec{x}'+d\hat{n}|}$$

Taylor expand this in 3-D

Recall in one dimension:

$$f(x+\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \overset{n^{\text{th}}\text{ derivative}}{f^{(n)}(x)} = f(x) + \varepsilon f'(x) + \frac{\varepsilon^2}{2} f''(x) + \dots$$

In three-dimensions: (Taylor series expansion about point $\vec{u}$) gradient of Ψ evaluated at $\vec{u}$

$$\Psi(\vec{x}+\vec{\varepsilon}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\vec{\varepsilon}\cdot\vec{\nabla})^n \Psi(\vec{x}) = \Psi(\vec{u}) + (\vec{\varepsilon}\cdot\overbrace{\vec{\nabla})\Psi(\vec{x})}^{\text{gradient of }\Psi\text{ evaluated at }\vec{u}} + \dots$$

$|\vec{\varepsilon}| \ll |\vec{u}|$

w.r.t. $\vec{x}'$ b/c $\vec{x}'$ is location of surface charge; $\vec{x}$ is fixed point

$$\Rightarrow \frac{1}{|(\vec{x}-\vec{x}')+d\hat{n}|} = \frac{1}{|\vec{x}-\vec{x}'|} + (d\hat{n}\cdot(-\vec{\nabla}_{\vec{x}'}))\frac{1}{|\vec{x}-\vec{x}'|} + \dots \quad ; d = d(\vec{x}')$$

$O(d^2)$

(See notes)

As the potential becomes:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma(\vec{x}')\,da'}{|\vec{x}-\vec{x}'|} - \left(\frac{1}{4\pi\varepsilon_0}\int_{S''} \sigma(\vec{x}')\left(\frac{1}{|\vec{x}-\vec{x}'|} - (d\hat{n}\cdot\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right)da''\right)$$

$$= (+)\frac{1}{4\pi\varepsilon_0}\int_{S''} \sigma(\vec{x}')\, d\hat{n}\cdot\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da''$$

can do integral over $\int da'$ as they are identical by assumption (no $\vec{x}''$ variables)

$$= (+)\frac{1}{4\pi\varepsilon_0}\int_{S'} \sigma(\vec{x}')\, d\hat{n}\cdot\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da'$$

$$\left[D(\vec{x}') = \sigma(\vec{x}')\cdot d(\vec{x}')\right]$$

$$= \underbrace{\frac{1}{4\pi\varepsilon_0}\int_{S'} D(\vec{x}')\,\hat{n}\cdot\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da'}$$

special case:

potential of a single point dipole at $\vec{x}'$ w/ dipole moment:

$$\vec{p}_{(\vec{x}')} = \hat{n}\, D(\vec{x}')\, da'.$$

- If single $\vec{p}$ point dipole at $\vec{x}'$ (only)

dummy variable

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \underbrace{\vec{p}(\vec{u})}_{\delta(\vec{u}-\vec{x}')}\cdot\vec{\nabla}_{\vec{u}}\left(\frac{1}{|\vec{x}-\vec{u}|}\right)d^3u$$

$$= \frac{1}{4\pi\varepsilon_0}\vec{p}(\vec{x}')\cdot\vec{\nabla}_{\vec{x}'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \checkmark$$

$$\left[\vec{\nabla}_{\vec{x}'}(\dots) = -\vec{\nabla}_{\vec{x}}(\dots)\right]$$

$$f = \frac{1}{\left|(\vec{x}-\vec{x}') + \underbrace{d\hat{n}}\right|} = \left[(x_1 - x_1' + d_1)^2 + (x_2 - x_2' + d_2)^2 + (x_3 - x_3' + d_3)^2\right]^{-1/2}$$

$$\vec{x} = x_1\hat{x} + x_2\hat{y} + x_3\hat{z}$$

$$\vec{x}' = x_1'\hat{x} + x_2'\hat{y} + x_3'\hat{z}$$

$$d\hat{n} = d_1\hat{x} + d_2\hat{y} + d_3\hat{z}$$

$$\frac{\partial f}{\partial x_1} = -\frac{1}{2}\left[\ \ldots\ \right]^{-3/2} \cdot 2(x_1 - x_1' + d_1) \cdot 1$$

but,

$$\frac{\partial f}{\partial x_1'} = -\frac{1}{2}\left[\ \ldots\ \right]^{-3/2} \cdot 2(x_1 - x_1' + d_1) \cdot (-1)$$

$$\Rightarrow \vec{\nabla}_{\vec{x}} f = -\vec{\nabla}_{\vec{x}'} f$$