Recall: elementary Lorentz force law: $\vec{F} = q\vec{v}\times\vec{B}$

$\uparrow$ charge q in external magnetic flux-density $\vec{B}$

If have a current density $\vec{J}(\vec{x})$ [at point $\vec{x}$] in external field $\vec{B}$:

$$\vec{J} = n(q\vec{v})$$

n: number density $[1/m^3]$ of charge carriers

$$[\vec{J}] = \frac{1}{m^3}\, C\, \frac{m}{s} = \frac{C}{s}\,\frac{1}{m^2} = \frac{A}{m^2} \checkmark$$

So in some infinitesimal d^3x:

$$d\vec{F} = dq\,\vec{v}\times\vec{B} \Rightarrow dq = nq\,d^3x \Rightarrow \quad \Rightarrow d\vec{F} = \vec{J}\,d^3x \times \vec{B}$$

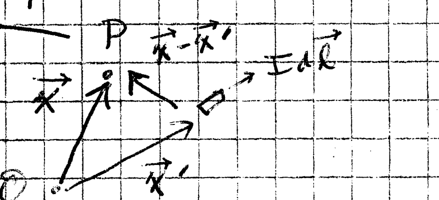
$$\Rightarrow \boxed{\vec{F} = \int d^3x\; \vec{J}\times\vec{B}} \qquad \begin{cases} \vec{J} = \vec{J}(\vec{x}) \\ \vec{B} = \vec{B}(\vec{x}) \end{cases}$$

$\Rightarrow$ At some point $\vec{x}$, the total magnetic torque about the origin O due to volume element d^3x will be:

$$d\vec{N} = \vec{x}\times d\vec{F} = \vec{x}\times(\vec{J}\,d^3x\times\vec{B})$$

$$\Rightarrow \boxed{\vec{N} = \int d^3x\; \vec{x}\times(\vec{J}\times\vec{B})}$$

Magnetostatic Differential Equations



We have:
(Biot-Savart) $$d\vec{B} = \frac{\mu_0 I}{4\pi}\cdot\frac{d\vec{l}\times\vec{x}}{|\vec{x}|^3}$$

More generally, for $\vec{B}$ at $\vec{x}$ due to current sources at $\vec{x}'$: $\quad I\,d\vec{l} = \vec{J}\,d^3x$

$$\Rightarrow \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\; \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

[Note: just like $\vec{E}(\vec{x}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int d^3x'\, \frac{\rho(\vec{x}')(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$]

Again, recall: $$\vec{\nabla}_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|} = -\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$\Rightarrow \vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\cdot\int d^3x'\; \vec{J}(\vec{x}')\times\vec{\nabla}_{\vec{x}}\frac{1}{|\vec{x}-\vec{x}'|}$$

using the vector identity:

$$\vec{\nabla}\times(\psi\vec{a}) = -\vec{a}\times\vec{\nabla}\psi + \psi\,\vec{\nabla}\times\vec{a} \;;$$

[In the integrand: $\vec{a} = \vec{J}(\vec{x}')$; $\psi = \dfrac{1}{|\vec{x}-\vec{x}'|}$]

$$\Rightarrow -a\times\vec{\nabla}\psi = \vec{\nabla}\times(\psi\vec{a}) - \psi(\vec{\nabla}\times\vec{a})$$

$$\Rightarrow -\vec{J}(\vec{x}') \times \vec{\nabla}_{x} \frac{1}{|\vec{x}-\vec{x}'|} = \vec{\nabla}_{x} \times \left(\frac{1}{|\vec{x}-\vec{x}'|} \vec{J}(\vec{x}') \right) - \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\vec{\nabla}_{x} \times \vec{J}(\vec{x}')}_{}$$

$= 0$ as $\vec{\nabla}$ is w.r.t. $\vec{x}$, but $\vec{J}$ is function of $\vec{x}'$!

$$\Rightarrow \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \vec{\nabla}_{x} \times \left(\frac{1}{|\vec{x}-\vec{x}'|} \vec{J}(\vec{x}') \right)$$

$$= \frac{\mu_0}{4\pi} \vec{\nabla}_{x} \times \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

As: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0 \quad \forall \vec{V}$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$$ (no magnetic charge)

Now, find $\vec{\nabla} \times \vec{B}$:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\frac{\mu_0}{4\pi} \vec{\nabla} \times \underbrace{\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}_{=\vec{V}} \right)$$

Using general identity: $\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}$

$$\Rightarrow \frac{\vec{\nabla} \times \vec{B}}{(\mu_0/4\pi)} = \vec{\nabla} \left[\vec{\nabla} \cdot \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right] - \vec{\nabla}^2 \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

[Again, $\vec{\nabla}$ is $\vec{\nabla}_{\vec{x}}$]

$$= \vec{\nabla} \left[\int d^3x' \, \vec{\nabla} \cdot \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \right] - \int d^3x' \, \vec{J}(\vec{x}') \vec{\nabla}^2 \frac{1}{|\vec{x}-\vec{x}'|}$$

(see proof)

$$= \vec{\nabla} \int d^3x' \, \vec{J}(\vec{x}') \cdot \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|} - \int d^3x' \, \vec{J}(\vec{x}') \, \vec{\nabla}^2 \frac{1}{|\vec{x}-\vec{x}'|}$$

Using: $\vec{\nabla}_{\vec{x}} \frac{1}{|\vec{x}-\vec{x}'|} = -\vec{\nabla}_{\vec{x}'} \frac{1}{|\vec{x}-\vec{x}'|}$, $\quad \vec{\nabla}^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi \delta(\vec{x}-\vec{x}')$

(see p. 16 of lecture notes) (see p. 19)

Let: $\vec{J}(\vec{x}') = J_x(\vec{x}')\hat{x} + J_y(\vec{x}')\hat{y} + J_z(\vec{x}')\hat{z}$.

$$\vec{\nabla}\cdot\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = (\partial_x\hat{x} + \partial_y\hat{y} + \partial_z\hat{z})\cdot\left[\frac{J_x(\vec{x}')\hat{x}}{|\vec{x}-\vec{x}'|} + \frac{J_y(\vec{x}')\hat{y}}{|\vec{x}-\vec{x}'|} + \frac{J_z(\vec{x}')\hat{z}}{|\vec{x}-\vec{x}'|}\right]$$

$$= J_x(\vec{x}')\,\partial_x\frac{1}{|\vec{x}-\vec{x}'|} + J_y(\vec{x}')\,\partial_y\frac{1}{|\vec{x}-\vec{x}'|} + J_z(\vec{x}')\,\partial_z\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \vec{J}\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \frac{\vec{\nabla}\times\vec{B}}{\mu_0/4\pi} = -\vec{\nabla}\int d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}_{\vec{x}'}\frac{1}{|\vec{x}-\vec{x}'|} + \int d^3x'\,\vec{J}(\vec{x}')\,4\pi\,\delta(\vec{x}-\vec{x}')$$

$$= -\vec{\nabla}\underbrace{\int_{\text{all space}} d^3x'\,\vec{J}(\vec{x}')\cdot\vec{\nabla}_{\vec{x}'}\frac{1}{|\vec{x}-\vec{x}'|}}_{\text{Integrate by parts, } u=\frac{1}{|\vec{x}-\vec{x}'|},\ \vec{v}=\vec{J}(\vec{x}')} + 4\pi\vec{J}(\vec{x})$$

$$(\text{Integral}) = \underbrace{\oint_S \frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}(\vec{x}')\cdot\hat{n}\,da'}_{\text{take } \vec{J}(\vec{x}')=0 \text{ at } \infty} - \int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{\nabla}'\cdot\vec{J}(\vec{x}')$$

[or for finite $\vec{x}$, $\frac{1}{|\vec{x}-\vec{x}'|}\to 0$ for $\vec{x}'$ at ∞]

$$\Rightarrow \frac{\vec{\nabla}\times\vec{B}}{\mu_0/4\pi} = +\vec{\nabla}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{\nabla}'\cdot\vec{J}(\vec{x}') + 4\pi\vec{J}(\vec{x})$$

But, recall continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

{ physical decrease in charge inside a small volume with time must correspond to a flow of charge out of the volume's surface, as charge is conserved

Magnetostatics: no change in net charge density anywhere in space $\Rightarrow \frac{\partial\rho}{\partial t}=0$

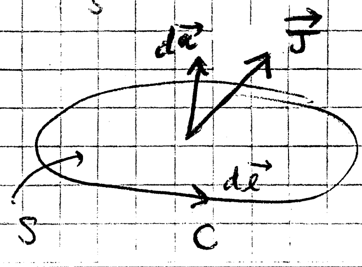
$\Rightarrow \vec{\nabla}\cdot\vec{J}=0$ in magnetostatics

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}\ \checkmark$$

over open surface!

Use Stokes' Theorem: $\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{a} = \oint_C \vec{V}\cdot d\vec{l}$

$$\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{a} = \mu_0\int_S \vec{J}\cdot d\vec{a} = \oint_C \vec{B}\cdot d\vec{l}$$



Clearly, $\int_S \vec{J}\cdot d\vec{a} = I$ (total current passing thru S)

$$\Rightarrow \boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I}$$ Ampere's Law

Magnetic Vector Potential

So far we have: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$\vec{\nabla}\cdot\vec{B} = 0$

- If $\vec{J}=0$ in region of interest, have $\vec{\nabla}\times\vec{B}=0$.
Then, just like $\vec{\nabla}\times\vec{E}=0 \Rightarrow \vec{E}=-\vec{\nabla}\Phi$, can define

$\vec{B} = -\vec{\nabla}\Phi_M$, "magnetic scalar potential"

$\Rightarrow \vec{\nabla}\cdot\vec{B} = -\vec{\nabla}\cdot\vec{\nabla}\Phi_M = -\vec{\nabla}^2\Phi_M = 0 \Rightarrow$ Laplace Equation as before:

$$\vec{\nabla}^2\Phi_M(\vec{x}) = 0$$

Also note:

As $\vec{\nabla}\cdot\vec{B}=0$ everywhere, it must be that: $\vec{B} = \vec{\nabla}\times\vec{A}$ (is curl of some vector field), $\vec{A}$: some vector field

And we had: $$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\Rightarrow \underbrace{\vec{A}(\vec{x})}_{\text{"magnetic vector potential"}} \equiv \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \underbrace{\vec{\nabla}\Psi(\vec{x})}_{\text{can arbitrarily add this "curl-free" term as}}$$

$\underbrace{\vec{\nabla}\times(\vec{\nabla}\Psi)=0}_{\text{does not affect physical quantity } \vec{B}(\vec{x})!}$

$\Rightarrow \vec{B}(\vec{x})$ is invariant under a "gauge transformation" of:

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi$$

Now if $\vec{B}=\vec{\nabla}\times\vec{A}$ substituted into: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$; and by vector identities:

$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0\vec{J}$

Because the only requirement on $\vec{A}$ is that: $\vec{\nabla}\times\vec{A}=\vec{B}$ } vector fields fully specified by their curl and divergence
we are free to choose $\vec{\nabla}\cdot\vec{A}$ however we like!

Simplest choice: $\vec{\nabla}\cdot\vec{A}=0$ ("Coulomb gauge")

$$\Rightarrow -\vec{\nabla}^2\vec{A} = \mu_0\vec{J} \quad \text{or} \quad \underbrace{\vec{\nabla}^2 A_i = -\mu_0 J_i}_{\text{Poisson Equation for each } i^{\text{th}} \text{ component of } A_i \text{ and } J_i}$$