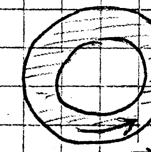
Example: Expansion of $\vec{A}$ in Multipoles [Jackson 5.8]

Localized cylindrically symmetric current distribution such that current flows only in the azimuthal direction. Current density function only of r and θ (in spherical coordinates):

$$\vec{J} = J(r,\theta)\,\hat{\phi}$$

Distribution is "hollow", in that there is a current-free region at the origin, as well as outside. "doughnut"

Find $\vec{A}$ inside and outside the current distribution:



spherical: (r',θ',ϕ')

In general, we have:

$\vec{J}(\vec{x}') = J(r',\theta')\,\hat{\phi}'$

no ϕ' dependence

$\vec{x}$: "observer point"

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi}\int [J(r',\theta')\,\hat{\phi}']\,\frac{1}{|\vec{x}-\vec{x}'|}\,d^3x'$$

Recalling:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{(2l+1)}\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$$Y_{lm}^*(\theta',\phi') = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta')\,e^{-im\phi'}$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,J(r',\theta')\,\hat{\phi}'\cdot \cancel{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{\cancel{(2l+1)}}\frac{r_<^l}{r_>^{l+1}}\cdot\sqrt{\frac{\cancel{2l+1}}{\cancel{4\pi}}}\sqrt{\frac{(l-m)!}{(l+m)!}}\cdot$$

$$\cdot P_l^m(\cos\theta')\,e^{-im\phi'}\cdot\sqrt{\frac{\cancel{2l+1}}{\cancel{4\pi}}}\sqrt{\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

$$= \frac{\mu_0}{4\pi}\sum_{l,m}\underbrace{\int (r')^2\,dr'\int d(\cos\theta')}_{\text{over } \vec{J}\neq 0}\int_0^{2\pi} d\phi'\,e^{-im\phi'}\,e^{im\phi}\cdot J(r',\theta')\,\frac{r_<^l}{r_>^{l+1}}\frac{(l-m)!}{(l+m)!}\cdot$$

$$\cdot P_l^m(\cos\theta')\,P_l^m(\cos\theta)\cdot\hat{\phi}'$$

Focus on ϕ' terms in ϕ'-integral:

$$\int_0^{2\pi} d\phi'\,\hat{\phi}'\,e^{im(\phi-\phi')} = \int_0^{2\pi} d\phi'\,\hat{\phi}'\left[\cos[m(\phi-\phi')] + i\sin[m(\phi-\phi')]\right]$$

not fixed in space w.r.t. ϕ'!

We can decompose $\hat{\phi}'$ into $(\hat{r},\hat{\theta},\hat{\phi})$ unit vectors as:

$$\hat{\phi}' = (\hat{\phi}'\cdot\hat{r})\,\hat{r} + (\hat{\phi}'\cdot\hat{\theta})\,\hat{\theta} + (\hat{\phi}'\cdot\hat{\phi})\,\hat{\phi}$$

In general: (in terms of cartesian coordinates)

remembering.....

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\ \hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \end{cases} \Rightarrow \hat{\phi}' = -\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y} \Rightarrow \text{using:}$$

OR

recall on: [HW #1]

$$\Rightarrow \hat{\phi}' = \overbrace{\left[-\sin\theta\cos\phi\sin\phi' + \sin\theta\sin\phi\cos\phi'\right]}^{=\hat{\phi}'\cdot\hat{r}}\hat{r} + \overbrace{\left[-\cos\theta\cos\phi\sin\phi' + \cos\theta\sin\phi\cos\phi'\right]}^{=\hat{\phi}'\cdot\hat{\theta}}\hat{\theta} + \overbrace{\left[\sin\phi\sin\phi' + \cos\phi\cos\phi'\right]}^{=\hat{\phi}'\cdot\hat{\phi}}\hat{\phi}$$

$$\boxed{\begin{aligned} \hat{x} &= \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi \\ \hat{y} &= \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi \\ \hat{z} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \end{aligned}}$$

$$= \sin\theta\cdot\sin[\phi-\phi']\,\hat{r} + \cos\theta\sin[\phi-\phi']\,\hat{\theta} + \cos(\phi-\phi')\,\hat{\phi}$$

For the terms in the ϕ'-integral:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\int_0^{2\pi} d\phi' \begin{pmatrix}\cos[m(\phi-\phi')] \\ +i\sin[m(\phi-\phi')]\end{pmatrix}\left(\sin\theta\sin[\phi-\phi']\,\hat{r} + \cos\theta\sin[\phi-\phi']\,\hat{\theta} + \cos[\phi-\phi']\,\hat{\phi}\right)$$

$$\left[\text{Recall: } \int_0^{2\pi} dx \sin mx\cos nx = 0\ \forall\ m,n \qquad \int_0^{2\pi} dx\cos mx\cos nx = \pi\delta_{mn} \qquad \int_0^{2\pi} dx\sin mx\sin nx = \pi\delta_{mn}\right]$$

$\Rightarrow$ only $m = \pm 1$ terms survive !! $\Rightarrow l \geq 1$

(see p. 104b) for additional work...,

So Now we have for $\vec{A}(\vec{x})$:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}2\pi\hat{\phi}\sum_{l=1}^{\infty}\int (r')^2 dr' \int d(\cos\theta')\, J(r',\theta')\frac{r_<^l}{r_>^{l+1}}\left[\frac{(l-1)!}{(l+1)!}P_l^1(\cos\theta')\cdot P_l^1(\cos\theta)\right]$$

$$\left[\text{Note: } \frac{(l-1)!}{(l+1)!} = \frac{1}{(l+1)l}\right]$$

$$= \hat{\phi}\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\underbrace{\int_0^{2\pi}d\phi}_{=2\pi}\cdot\int d(\cos\theta')\int (r')^2 dr'\, J(r',\theta')\frac{r_<^l}{r_>^{l+1}}\frac{1}{(l+1)l}P_l^1(\cos\theta')P_l^1(\cos\theta)$$

Recall: (trick)

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \sin\theta(-\cos\theta)\Big|_0^{2\pi} + \int_0^{2\pi} \cos\theta\cos\theta\, d\theta$$

$u = \sin\theta$
$dv = \sin\theta\, d\theta$

$$= 0 + \int_0^{2\pi} (1-\sin^2\theta)\, d\theta$$

$$= 2\pi - \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$\Rightarrow 2\int_0^{2\pi} \sin^2\theta\, d\theta = 2\pi \Rightarrow \int_0^{2\pi} \sin^2\theta\, d\theta = \pi \checkmark$$

Writing in terms of $m = \pm 1$ terms only:
The expression for $\vec{A}(\vec{x})$ becomes:

$\boxed{\text{Note: } \cos(-x) = \cos x,\ \sin(-x) = -\sin x}$
useful for $m = -1$

$$= \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int (r')^2 dr' \int d(\cos\theta') \Bigg[\overbrace{\left(i\pi\sin\theta\,\hat{r} + i\pi\cos\theta\,\hat{\theta} + \pi\hat{\phi}\right)\cdot\frac{(l-1)!}{(l+1)!}\cdot}^{m=+1}$$

$$P_l^{+1}(\cos\theta)P_l^{+1}(\cos\theta') \quad +$$

$m=-1$: $$\left(-i\pi\sin\theta\,\hat{r} - i\pi\cos\theta\,\hat{\theta} + \pi\hat{\phi}\right)\cdot\frac{(l+1)!}{(l-1)!}\cdot$$

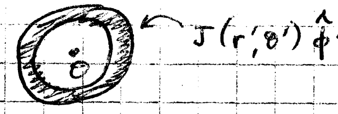
$$P_l^{-1}(\cos\theta)\,P_l^{-1}(\cos\theta')\Bigg]\cdot J(r',\theta')\frac{r_<^l}{r_>^{l+1}}$$

$$\left(\text{Recall: } P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \quad \Rightarrow P_l^{-1}(x) = -\frac{(l-1)!}{(l+1)!}P_l^1(x)\right)$$

$$\Rightarrow P_l^{-1}(\cos\theta)P_l^{-1}(\cos\theta') = \left[\frac{(l-1)!}{(l+1)!}\right]^2\cdot P_l^1(\cos\theta)\cdot P_l^1(\cos\theta')$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int (r')^2 dr' \int d(\cos\theta')\left[2\pi\hat{\phi}\, P_l^1(\cos\theta)\,P_l^1(\cos\theta')\right]\cdot\frac{(l-1)!}{(l+1)!}\cdot$$

$$\cdot J(r',\theta')\cdot\frac{r_<^l}{r_>^{l+1}} \checkmark$$



Inside the "doughnut", $r_< = r$, $r_> = r'$:

$$\vec{A}_{in}(\vec{x}) = \hat{\phi}\,\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int_0^{2\pi} d\phi' \int d(\cos\theta') \int dr'\, \frac{r^l}{(r')^{l+1}}\,\frac{1}{(l+1)\,l}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta) \quad \checkmark$$

Outside the "doughnut", $r_< = r'$, $r_> = r$:

$$\vec{A}_{out}(\vec{x}) = \hat{\phi}\,\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int_0^{2\pi} d\phi' \int d(\cos\theta') \int dr'\, \frac{(r')^{l+2}}{r^{l+1}}\,\frac{1}{(l+1)\,l}\,P_l^1(\cos\theta')\,P_l^1(\cos\theta) \quad \checkmark$$

Could find: $\vec{B}_{in}(\vec{x}) = \vec{\nabla}\times\vec{A}_{in}$ $\qquad \vec{B}_{out}(\vec{x}) = \vec{\nabla}\times\vec{A}_{out}$

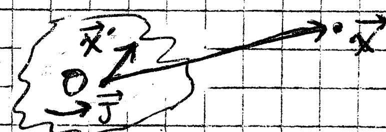
Magnetic Fields of Localized Current Density

Consider "localized" current distribution, "small"? relative to scale of observer.

Complete treatment: vector spherical harmonics, discussed later (Phy 613) when treating multipole radiation.

Now, just lowest-order:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$



As shown on pp. 86b – 86c of the lecture notes, for $|\vec{x}'| \ll |\vec{x}|$, we can expand $\frac{1}{|\vec{x}-\vec{x}'|}$ as:

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots$$

[just from expansion (Taylor 3D) in $\vec{x}'$: $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \left(\vec{\nabla}_{x'}\frac{1}{|\vec{x}-\vec{x}'|}\right)\cdot\vec{x}' + \ldots$]

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\vec{J}(\vec{x}')\left[\frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + \ldots\right]$$

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\int d^3x'\,\vec{J}(\vec{x}') + \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int d^3x'\,\vec{J}(\vec{x}')\left[\vec{x}\cdot\vec{x}'\right]$$

$$\left(\text{or}\quad A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\int d^3x'\,J_i(\vec{x}') + \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int d^3x'\,J_i(\vec{x}')\left[\vec{x}\cdot\vec{x}'\right]\right)$$

Consider the first term; use $\int_V \vec{\nabla}u\cdot\vec{V}\,d^3x = \oint_S u\vec{V}\cdot\hat{n}\,da - \int_V u(\vec{\nabla}\cdot\vec{V})\,d^3x$

$$\underbrace{\int_V d^3x'\,J_i(\vec{x}')} = \oint_{S\text{-bounding } V \text{ (where } \vec{J}\to 0\text{), localized/finite!}} x_i'\,\vec{J}(\vec{x}')\cdot\hat{n}\,da' - \int_V x_i'\underbrace{\left(\vec{\nabla}\cdot\vec{J}(\vec{x}')\right)}_{=0} d^3x'$$

Trick:

$J_x = \vec{\nabla}x \cdot \vec{J}$ $\quad\Rightarrow\quad u = x_i'$, $\vec{V} = \vec{J}(\vec{x}')$ $\Rightarrow$ $\vec{\nabla}u\cdot\vec{V} = J_i$

$= \hat{x}\cdot\vec{J} = J_x$