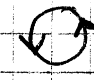
$$\Rightarrow \int d^3x' \, J_i(\vec{x}') = 0$$

[Makes sense: think about a ring of current!]



Clearly, $\int J_x \, d^3x = 0$!

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|}\underbrace{\cancel{\int d^3x' \, J_i(\vec{x}')}}_{0} + \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int d^3x' \, J_i(\vec{x}')(\vec{x}\cdot\vec{x}')$$

analogous to "monopole" term in electrostatics:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \dots\right)$$

$\Rightarrow$ no "monopole" term in magnetostatics (as expected!!) no magnetic charge!!

Work on second integral in expansion:

i^{th} component:

$$\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\int d^3x' \, \underbrace{\vec{J}(\vec{x}')}_{\text{vector}}\underbrace{(\vec{x}\cdot\vec{x}')}_{\text{scalar}} \Rightarrow \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\vec{x}\cdot\int d^3x' \, J_i(\vec{x}')\,\vec{x}'$$

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3} x_j \underbrace{\int d^3x' \, J_i(\vec{x}')\, x_j'}$$ (in index notation)

• $$\int_V d^3x' \, J_i(\vec{x}')\, x_j' = \int_V d^3x' \left[\vec{\nabla} x_i' \cdot x_j' \vec{J}(\vec{x}')\right]$$

$u = x_i'$ $\quad \vec{v} = x_j' \vec{J}(\vec{x}')$

$$= \oint_S \underbrace{\cancel{x_i' x_j' \vec{J}(\vec{x}')}}_{0 \text{ at } \infty} \cdot \hat{n}\, da' - \int_V d^3x' \, x_i' \underbrace{\left[\vec{\nabla}'\cdot x_j' \vec{J}(\vec{x}')\right]}$$

use: $\vec{\nabla}\cdot(\psi\vec{a}) = \vec{a}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{a}$

$$= -\int_V d^3x' \, x_i' \left[\underbrace{\vec{J}(\vec{x}')\cdot\vec{\nabla}' x_j'}_{= J_j(\vec{x}')} + x_j' \underbrace{\vec{\nabla}'\cdot\vec{J}(\vec{x}')}_{=0}\right]$$

$$= -\int_V d^3x \, x_i' \, J_j(\vec{x}')$$

$$\Rightarrow \int_V d^3x' \, J_i(\vec{x}')\, x_j' = -\int_V d^3x' \, J_j(\vec{x}')\, x_i'$$

So now we can write: $\int d^3x' \, J_i(\vec{x}') x_j' = \int d^3x' \left[J_i x_j' + \overbrace{J_i x_j' + J_j x_i'}^{=0}\right]$

$$\Rightarrow 2\int d^3x' \left[J_i x_j'\right] = \int d^3x' \left[J_i x_j' - J_j x_i'\right]$$

$$\Rightarrow A_i(\vec{x}) = \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3} x_j \int d^3x' \, J_i(\vec{x}')x_j'$$

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3} x_j \cdot \frac{1}{2}\int d^3x' \underbrace{\left[J_i x_j' - J_j x_i'\right]}$$ (j: repeated index) i: fixed index

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3} \frac{1}{2}\int d^3x' \left\{J_i(\vec{x}')\left[\vec{x}\cdot\vec{x}'\right] - x_i'\left[\vec{x}\cdot\vec{J}(\vec{x}')\right]\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\cdot\frac{1}{2}\int d^3x' \left\{\vec{x}\times\left(\vec{J}(\vec{x}')\times\vec{x}'\right)\right\}_i$$

use: $\left[\vec{A}\times(\vec{B}\times\vec{C})\right]_i = (\vec{A}\cdot\vec{C})B_i - (\vec{A}\cdot\vec{B})C_i$

$\vec{A}=\vec{x}, \vec{B}=\vec{J}, \vec{C}=\vec{x}'$

flip

flip

customary:

$$= +\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}\cdot\frac{1}{2}\left[\left(\int d^3x' \left[\vec{x}'\times\vec{J}(\vec{x}')\right]\right)\times\vec{x}\right]$$

$$\Rightarrow \vec{A}(\vec{x}) \equiv \frac{\mu_0}{4\pi}\cdot\frac{1}{|\vec{x}|^3}\cdot(\vec{m}\times\vec{x}) + \ldots$$

$$\vec{m} \equiv \frac{1}{2}\int d^3x' \left[\vec{x}'\times\vec{J}(\vec{x}')\right]$$ "magnetic moment" of current density $\vec{J}$ (fixed vector)

$$\vec{M} \equiv \frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$$ "magnetization" (magnetic moment density)

- $\vec{m}$ is lowest non-vanishing term in expansion of $\vec{A}(\vec{x})$ for localised magnetostatic current distribution. $\vec{m} = \int d^3x' \vec{M}(\vec{x}')$

What is $\vec{B}$? $\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})$

$$\vec{B} = \vec{\nabla}\times\frac{\mu_0}{4\pi}\frac{1}{|\vec{x}|^3}(\vec{m}\times\vec{x}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left(\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}\right)$$

use: $\vec{\nabla}\times(\vec{a}\times\vec{b}) = \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a}) + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b}$

$\vec{a} = \frac{\vec{m}}{|\vec{x}|^3}, \ \vec{b} = \vec{x}$

$$\Rightarrow \frac{\vec{B}(\vec{x})}{(\mu_0/4\pi)} = \frac{\vec{m}}{|\vec{x}|^3}(\vec{\nabla}\cdot\vec{x}) - \vec{x}\left(\vec{\nabla}\cdot\frac{\vec{m}}{|\vec{x}|^3}\right) - \left(\frac{\vec{m}}{|\vec{x}|^3}\cdot\vec{\nabla}\right)\vec{x} + (\vec{x}\cdot\vec{\nabla})\frac{\vec{m}}{|\vec{x}|^3}$$

- $\vec{\nabla}\cdot\vec{x} = 3$
- $\vec{\nabla}\cdot\frac{\vec{m}}{|\vec{x}|^3} = \partial_x \frac{m_x}{(x^2+y^2+z^2)^{3/2}} + \ldots = m_x\cdot\left(-\frac{3}{\cancel{2}}\right)\frac{1}{|\vec{x}|^5}\cdot\cancel{2}x + \ldots = \frac{-3m_x\cdot x}{|\vec{x}|^5} + \ldots$

$$\Rightarrow \vec{\nabla}\cdot\frac{\vec{m}}{|\vec{x}|^3} = \frac{-3\vec{m}\cdot\vec{x}}{|\vec{x}|^5}$$

- $\left(\frac{\vec{m}}{|\vec{x}|^3}\cdot\vec{\nabla}\right)\vec{x} = \left(\frac{m_x\partial_x}{|\vec{x}|^3} + \ldots\right)(x\hat{x} + y\hat{y} + z\hat{z}) = \frac{m_x}{|\vec{x}|^3}\hat{x} + \ldots = \frac{\vec{m}}{|\vec{x}|^3}$
- $(\vec{x}\cdot\vec{\nabla})\frac{\vec{m}}{|\vec{x}|^3} = (x\partial_x + \ldots)\left(\frac{\vec{m}}{|\vec{x}|^3}\right) =$

$$= \vec{m}(x\partial_x + \ldots)\frac{1}{|\vec{x}|^3} = \vec{m}\left[x\cdot\left(-\frac{3}{\cancel{2}}\right)\frac{1}{|\vec{x}|^5}\cancel{2}x + \ldots\right]$$

$$= \vec{m}\left[\frac{-3x^2}{|\vec{x}|^5} - \frac{3y^2}{|\vec{x}|^5} - \frac{3z^2}{|\vec{x}|^5}\right]$$

$$= \vec{m}\left(\frac{-3}{|\vec{x}|^5}\cdot|\vec{x}|^2\right) = \frac{-3\vec{m}}{|\vec{x}|^3}$$

$$\Rightarrow \frac{\vec{B}(\vec{x})}{\mu_0/4\pi} = \frac{\vec{m}}{|\vec{x}|^3}\cdot 3 + \vec{x}\,\frac{3\vec{m}\cdot\vec{x}}{|\vec{x}|^5} - \frac{\vec{m}}{|\vec{x}|^3} + \frac{-3\vec{m}}{|\vec{x}|^3}$$

$$= \frac{1}{|\vec{x}|^3}\left[\cancel{3\vec{m}} - \cancel{3\vec{m}} + \frac{(3\vec{m}\cdot\cancel{|\vec{x}|}\hat{n})\cancel{|\vec{x}|}\hat{n}}{\cancel{|\vec{x}|^2}} - \frac{\vec{m}}{1}\right]$$

$$= \boxed{\frac{1}{|\vec{x}|^3}\left[3(\vec{m}\cdot\hat{n})\hat{n} - \vec{m}\right]}, \quad \hat{n} \equiv \frac{\vec{x}}{|\vec{x}|}$$

; or if $\vec{m}$ at $\vec{x_0}$,

$$\hat{n} = \frac{\vec{x}-\vec{x_0}}{|\vec{x}-\vec{x_0}|}$$

$$|\vec{x}| \to |\vec{x}-\vec{x_0}|$$

compare to: (dipole)

$$\frac{\vec{E}(\vec{x})}{4\pi\varepsilon_0} = \frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}|^3}$$

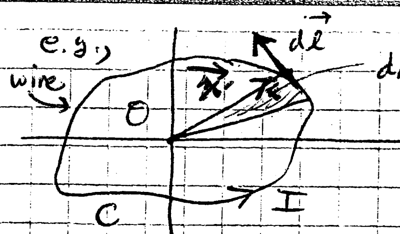
✓

Special Case:

Note: if current confined to a plane (wire in a plane)

$\vec{J}(\vec{x})\, d^3x = I\, d\vec{l}$ along C

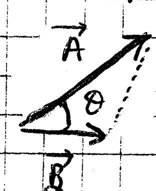
e.g.,



$$\Rightarrow \vec{m} = \frac{1}{2}\int d^3x'\, \vec{x}' \times \vec{J}(\vec{x}') = \frac{1}{2}\int_C \vec{x}' \times I\, d\vec{l} = \frac{I}{2}\int \vec{x}' \times d\vec{l}$$

$\rightarrow$ $\vec{m}$ is $\perp$ to plane of loop!

Recall from vector geometry:



$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$
$$= 2\,(\text{area of triangle bounded by } \vec{A} \,\&\, \vec{B})$$

$$\Rightarrow |\vec{m}| = I \cdot \frac{1}{2}\, 2 \int da = I \cdot (\text{area})$$

$\uparrow$ independent of "shape" of C!

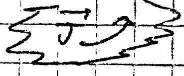
Spherical Volume Integral of $\vec{B}$

Recall: When discussing $\int_V \vec{E}(\vec{x})\, d^3x$ over some spherical volume, found we had to modify $\vec{E}$-field of a dipole so that if the sphere contains $\vec{p}$:

$$\int \vec{E}(\vec{x})\, d^3x = -\frac{\vec{p}}{3\epsilon_0} \Rightarrow \vec{E}_{\text{dipole}}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{n}(\vec{p}\cdot\hat{n}) - \vec{p}}{|\vec{x}-\vec{x}_0|^3} - \frac{4\pi}{3}\vec{p}\,\delta(\vec{x}-\vec{x}_0)\right]$$

(see pp. 88–92 of lecture notes)

Now:



consider: sphere containing $\vec{J}$ or not (radius R)

$$\int_{V: r<R} \vec{B}(\vec{x})\, d^3x = \int_{V: r<R} (\vec{\nabla}\times\vec{A})\, d^3x$$

Consider, e.g., the $\hat{x}$-component:

$$\int_V B_x(\vec{x})\, d^3x = \int_V (\vec{\nabla}\times\vec{A})\cdot\hat{x}\; d^3x$$

use: $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$

$$\Rightarrow \vec{\nabla}\cdot(\vec{A}\times\hat{x}) = \hat{x}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot\underbrace{(\vec{\nabla}\times\hat{x})}_{=0} \Rightarrow (\vec{\nabla}\times\vec{A})\cdot\hat{x} = \vec{\nabla}\cdot(\vec{A}\times\hat{x})$$