$$\Rightarrow \int_V (\vec\nabla\times\vec A)\cdot\hat x\, d^3x = \int_V d^3x\, \vec\nabla\cdot(\vec A\times\hat x) = \oint_S \underbrace{(\vec A\times\hat x)\cdot\hat n}_{=\hat x\cdot(\hat n\times\vec A)}\, da$$ (110)

[by vector identities $(\vec a\times\vec b)$ $=\vec b\cdot($

$\Rightarrow$ true for the $\hat x$, $\hat y$, and $\hat z$ components:
$$\int_V (\vec\nabla\times\vec A)\, d^3x = \oint_S (\hat n\times\vec A)\, da = \oint_S R^2\, d\Omega\, (\hat n\times\vec A)$$

Now recalling: $\vec A(\vec x) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec J(\vec x')}{|\vec x-\vec x'|}$

$$\Rightarrow R^2\oint_S d\Omega(\hat n\times\vec A) = -R^2\oint_S d\Omega(\vec A\times\hat n) = -R^2\oint_S d\Omega\left(\left[\frac{\mu_0}{4\pi}\int d^3x' \frac{\vec J(\vec x')}{|\vec x-\vec x'|}\right]\times\hat n\right)$$

$$= -\frac{\mu_0R^2}{4\pi}\int_V d^3x'\, \vec J(\vec x')\times\underbrace{\oint_S d\Omega\frac{\hat n}{|\vec x-\vec x'|}}$$ (interchanging order of primed and unprimed integrations)

Recalling: $$\oint_S d\Omega\frac{\hat n}{|\vec x-\vec x'|} = \frac{4\pi}{3}\frac{r_<}{r_>^2}\hat n'$$

(long, complicated proof from before, pp. 89-90)

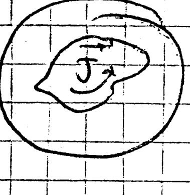
$$\Rightarrow = -\frac{\mu_0R^2}{\cancel{4\pi}}\int_V d^3x'\, \vec J(\vec x')\times\frac{\cancel{4\pi}}{3}\frac{r_<}{r_>^2}\hat n'$$

with $\hat n' = \frac{\vec x'}{|\vec x'|} = \frac{\vec x'}{r'}$

$$= -\frac{\mu_0}{3}\int_V d^3x'\, \vec J(\vec x')\times\frac{r_<R^2}{r_>^2}\cdot\frac{\vec x'}{r'}$$

$$= +\frac{\mu_0}{3}\int_V d^3x' \left[\vec x'\times\vec J(\vec x')\right]\left(\frac{r_<R^2}{r_>^2 r'}\right)$$ $r_<$, $r_>$ of r' and R as $\vec x$ is from $\oint_S$

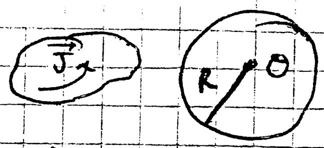
If $\vec J$ enclosed within sphere:



$r_< = r'$
$r_> = R$
$$\Rightarrow \int_V \vec B(\vec x)\, d^3x = +\frac{\mu_0}{3}\int d^3x' \left[\vec x'\times\vec J(\vec x')\right]\cdot\frac{r'R^2}{R^2r'}$$

$$= +\frac{\mu_0}{3}\underbrace{\int d^3x' \left[\vec x'\times\vec J(\vec x')\right]}_{=2\vec m} = \frac{\mu_0}{3}\cdot 2\vec m$$

If opposite case:



$r_< = R, \quad r_> = r'$

$$\int_V \vec{B}\, d^3x = \frac{\mu_0}{3}\int d^3x' \left[\vec{x}' \times \vec{J}(\vec{x}')\right] \frac{R\, R^2}{(r')^2 (r')} = \frac{\mu_0}{3}\int d^3x' \underbrace{\left[\vec{x}' \times \vec{J}(\vec{x}')\right]}_{=\left[\vec{J}(\vec{x}') \times (-\vec{x}')\right]} \frac{R^3}{(r')^3}$$

$$\Rightarrow \left[\text{recall: } \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right]$$

with $\vec{x} = \vec{0}$,

$$\Rightarrow \; = \frac{\cancel{\mu_0}}{3} R^3 \cdot \frac{4\pi \vec{B}(0)}{\cancel{\mu_0}} = \frac{4}{3}\pi R^3 \vec{B}(0) \; ✓$$

For magnetic dipole, $\vec{B}_{dipole}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}$ same form as $\vec{E}_{dipole}$ $(\vec{p} \to \vec{m})$

and so $\int_V \vec{B}_{dipole}(\vec{x})\, d^3x = 0$ by convention,

so to satisfy $\int_V \vec{B}(\vec{x})\, d^3x = \frac{2}{3}\mu_0 \vec{m}$ must re-define:

$$\vec{B}_{dipole}(\vec{x}) = \left[\frac{\mu_0}{4\pi}\,\frac{3\hat{n}(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3} + \frac{2}{3}\mu_0 \vec{m}\,\delta(\vec{x})\right] ✓$$

Magnetic
Force and Torque

Recall general expression (p. 99):

$$\vec{F} = \int d^3x' \vec{J}(\vec{x}') \times \vec{B}(\vec{x}')$$; total magnetic force on a current distribution in external $\vec{B}$-field

Taylor expand $\vec{B}(\vec{x}')$ about $\vec{x} = 0$:

$$\underbrace{B_k(\vec{x}')}_{k^{th}\text{ component}} = B_k(0) + \vec{x}' \cdot \vec{\nabla} B_k(0) + \ldots$$

$\vec{x}'$ is fixed point of $\vec{J}$ sources
$\vec{x}$: is "coordinate system" variable

$$\Rightarrow \underbrace{F_i}_{i^{th}\text{ component}} = \varepsilon_{ijk}\int d^3x' J_j(\vec{x}')\, B_k(\vec{x}')$$

$$= \varepsilon_{ijk}\int d^3x'\, J_j(\vec{x}')\left[B_k(0) + \vec{x}'\cdot\vec{\nabla} B_k(0) + \ldots\right]$$

$$= \varepsilon_{ijk} B_k(0)\int d^3x'\, J_j(\vec{x}') + \varepsilon_{ijk}\int d^3x'\, J_j(\vec{x}')\, \vec{x}'\cdot\vec{\nabla} B_k(0) + \ldots$$