But we showed, see pp. 105-106, that: $\int d^3x'\, \vec{J}(\vec{x}') = 0$

To lowest order,

$$F_i = \varepsilon_{ijk} B_k(0) \underbrace{\int d^3x'\, J_j(\vec{x}')}_{0} + \varepsilon_{ijk} \int d^3x'\, J_j(\vec{x}')\, \vec{x}' \cdot \vec{\nabla} B_k(0)$$

$$= \varepsilon_{ijk}\, \vec{\nabla} B_k(0) \cdot \int d^3x'\, J_j(\vec{x}')\, \vec{x}'$$

Introduce index ℓ:

$$= \varepsilon_{ijk} \left[\vec{\nabla} B_k(0)\right]_\ell \int d^3x'\, J_j(\vec{x}')\, x'_\ell$$

$$= \varepsilon_{ijk} \left[\partial_\ell B_k(0)\right] \cdot \int d^3x'\, J_j(\vec{x}')\, x'_\ell$$

[Recall: (p. 106) $\int d^3x'\, J_j(\vec{x}')\, x'_\ell = \frac{1}{2} \int d^3x' \left[J_j(\vec{x}')\, x'_\ell - J_\ell(\vec{x}')\, x'_j\right]$]

$$= \varepsilon_{ijk} \left[\partial_\ell B_k(0)\right] \frac{1}{2} \int d^3x' \left[J_j(\vec{x}')\, x'_\ell - J_\ell(\vec{x}')\, x'_j\right]$$

$$= \varepsilon_{ijk} \frac{1}{2} \int d^3x' \left[\left(\vec{\nabla} B_k(0) \cdot \vec{x}'\right) J_j(\vec{x}') - \left(\vec{\nabla} B_k(0) \cdot \vec{J}(\vec{x}')\right) x'_j\right]$$

[use: $B_j(\vec{A}\cdot\vec{C}) - C_j(\vec{A}\cdot\vec{B}) = [\vec{A} \times (\vec{B} \times \vec{C})]_j = \varepsilon_{jnp} A_n (\vec{B} \times \vec{C})_p$]

$$= \varepsilon_{ijk} \frac{1}{2} \int d^3x' \left[\vec{\nabla} B_k(0) \times \left(\vec{J}(\vec{x}') \times \vec{x}'\right)\right]_j$$

$$= \varepsilon_{ijk} \cdot \frac{1}{2} \cdot \int d^3x'\, \varepsilon_{jnp} \left(\vec{\nabla} B_k(0)\right)_n \left(\vec{J}(\vec{x}') \times \vec{x}'\right)_p$$

$$= \varepsilon_{ijk} \cdot \varepsilon_{jnp} \cdot \underbrace{\frac{1}{2} \cdot \int d^3x' \left(\vec{J}(\vec{x}') \times \vec{x}'\right)_p}_{= -m_p} \partial_n B_k(0)$$

$$\left[\vec{m} = \frac{1}{2} \int d^3x' \left[\vec{x}' \times \vec{J}(\vec{x}')\right]\right]$$

$$= \varepsilon_{ijk}\, \varepsilon_{jnp}\, \partial_n B_k(0)\, (-m_p)$$

$$= \varepsilon_{ijk}\, \varepsilon_{jpn}\, m_p\, \partial_n B_k(0)$$

$$= \varepsilon_{ijk} (\vec{m} \times \vec{\nabla})_j B_k(0)$$

$$= \left[(\vec{m} \times \vec{\nabla}) \times \vec{B}\right]_i$$

$\Rightarrow$ to lowest order,

see p. 112b

$$\underbrace{\vec{F} = (\vec{m} \times \vec{\nabla}) \times \vec{B} = \vec{\nabla}(\vec{m} \cdot \vec{B})}_{\text{total force on dipole } \vec{m} \text{ in external } \vec{B}}$$

or; $F_i = \left[(\vec{m} \times \vec{\nabla}) \times \vec{B}\right]_i = \varepsilon_{ijk} (\vec{m} \times \vec{\nabla})_j B_k$

$= \varepsilon_{ijk} \left[\varepsilon_{jpq} m_p \nabla_q\right] B_k = \varepsilon_{ijk} \varepsilon_{jpq} m_p \nabla_q B_k$

$= \varepsilon_{jki} \varepsilon_{jpq} m_p \nabla_q B_k$

$= (\delta_{kp} \delta_{iq} - \delta_{kq} \delta_{ip}) m_p \nabla_q B_k$ over $\int d^3x'$

constant result of doing an integral

$= m_k \nabla_i B_k - m_i \nabla_k B_k = \nabla_i (\vec{m} \cdot \vec{B}) - m_i \underbrace{(\vec{\nabla} \cdot \vec{B})}_{=0}$

$\Rightarrow \vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$

Also, recall for the total torque on a current $\vec{J}$ in a field $\vec{B}$:

$$\vec{N} = \int d^3x' \, \vec{x}' \times \left(\vec{J}(\vec{x}') \times \vec{B}(\vec{x}')\right)$$

Expanding: $B_k(\vec{x}') = B_k(0) + \vec{x}' \cdot \nabla B_k(0) + \ldots$

to very lowest order: $\vec{B}(\vec{x}') \cong \vec{B}(0)$

$$\vec{N} \cong \int d^3x' \, \vec{x}' \times \left[\vec{J}(\vec{x}') \times \vec{B}(0)\right]$$

$$= \int d^3x' \left[(\vec{x}' \cdot \vec{B}(0))\vec{J}(\vec{x}') - (\vec{x}' \cdot \vec{J}(\vec{x}'))\vec{B}(0)\right]$$

$$= \underbrace{\int d^3x' \, \vec{J}(\vec{x}')\,(\vec{x}' \cdot \vec{B}(0))}_{1} - \underbrace{\int d^3x' \, \vec{B}(0)\, \vec{x}' \cdot \vec{J}(\vec{x}')}_{2}$$

1 $\Rightarrow$

$$N_i = \int d^3x' \, \underbrace{J_i x_j'}\, B_j(0)$$

$$= \int d^3x' \, \tfrac{1}{2}\left[J_i x_j' - J_j x_i'\right] B_j(0)$$

$$= \tfrac{1}{2}\int d^3x' \left[J_i (\vec{B}(0) \cdot \vec{x}') - x_i' (\vec{B}(0) \cdot \vec{J})\right]$$

use: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

$$= \tfrac{1}{2}\int d^3x' \left[\vec{B}(0) \times (\vec{J}(\vec{x}') \times \vec{x}')\right]_i$$

$$= \left[\vec{B}(0) \times (-\vec{m})\right]_i = \left[\vec{m} \times \vec{B}(0)\right]_i$$

2 $$\int d^3x' \, \vec{B}(0)\, \vec{x}' \cdot \vec{J}(\vec{x}') = \vec{B}(0) \int d^3x' \, \vec{x}' \cdot \vec{J}(\vec{x}')$$

$$= \vec{B}(0) \cdot \underbrace{\int d^3x' \, x_j' J_j}$$

Previously showed (p. 106) that: $\int d^3x' \, J_i x_j' = -\int d^3x' \, J_j x_i'$

So if $i = j$, only possible if $\underline{\underline{\int d^3x' \, x_j' J_j = 0}}$

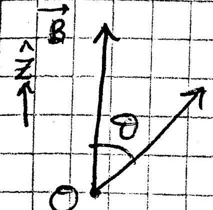
To lowest order, $\vec{N} = \vec{m} \times \vec{B}(0)$ for dipole $\vec{m}$ at $\vec{x} = 0$

Note: we have $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$. Recall, for conservative force, $\vec{F} = -\vec{\nabla}\mathcal{U}$,

$\Rightarrow \mathcal{U} = -\vec{m}\cdot\vec{B}$ $\Rightarrow$ dipole $\vec{m}$ will be parallel to $\vec{B}$ to minimize the energy

$\mathcal{U}$ = potential energy

"Application": Larmor Precession, Magnetic Resonance



$\vec{m} = \vec{\mu} = \gamma\hbar\vec{J}$ γ: "gyromagnetic ratio"

$\vec{J}$: spin (dimensionless), $\vec{L} = \hbar\vec{J}$

Note: in general, $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$, if $\vec{B} = B_0\hat{z}$ uniform, $\vec{\nabla}(\vec{m}\cdot\vec{B}) = 0$.

But $\vec{N} = \vec{m}\times\vec{B} = \frac{d}{dt}\vec{L}$

Since $\vec{F}=0$, $\vec{m}$ remains fixed at the origin, and $\vec{m} = \gamma\hbar J\,\hat{r}$

$$\vec{N} = \vec{m}\times\vec{B} = \gamma\hbar J\hat{r}\times B_0\hat{z} = \gamma\hbar J\hat{r}\times\left[B_0\cos\theta\,\hat{r} - B_0\sin\theta\,\hat{\theta}\right]$$

$$= -\gamma\hbar J B_0\sin\theta\,\hat{\phi} \Rightarrow \vec{N}\perp \text{ plane of } \vec{B} \text{ and } \vec{m} \text{ in } \hat{\phi}\text{-direction}$$

- no $\vec{N}$ in $\hat{\theta}$ direction, so angle θ does not change

$$\Rightarrow -\gamma\hbar J B_0\sin\theta\,\hat{\phi} = \frac{d}{dt}\vec{L}$$

$$\frac{d}{dt}(L_x) = \frac{d}{dt}(\hbar J_x) = \frac{d}{dt}(\hbar J\sin\theta\cos\phi)$$
$$= -\hbar J\sin\theta\sin\phi\frac{d\phi}{dt}$$

$$\frac{d}{dt}(L_y) = \frac{d}{dt}(\hbar J_y) = \frac{d}{dt}(\hbar J\sin\theta\sin\phi)$$
$$= \hbar J\sin\theta\cos\phi\frac{d\phi}{dt}$$

$$\frac{d}{dt}(L_z) = \frac{d}{dt}(\hbar J_z) = \frac{d}{dt}(J\cos\theta) = 0$$

and
$$\frac{d}{dt}(\vec{J}\cdot\vec{J}) = 2\vec{J}\cdot\frac{d\vec{J}}{dt} = 2\vec{J}\cdot\frac{1}{\hbar}(\vec{m}\times\vec{B}) = 2\frac{\vec{m}}{\gamma\hbar}\cdot\frac{1}{\hbar}(\vec{m}\times\vec{B}) = 0$$

$\Rightarrow |\vec{J}|^2$ is constant

- J is constant

$$\vec{N} = -\gamma\hbar J B_0\sin\theta(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}) = \frac{d}{dt}\vec{L}$$

$$\Rightarrow \hat{x}:\ \gamma\cancel{\hbar}\cancel{J}B_0\cancel{\sin\theta}\cancel{\sin\phi} = -\cancel{\hbar}\cancel{J}\cancel{\sin\theta}\cancel{\sin\phi}\frac{d\phi}{dt} \Rightarrow \frac{d\phi}{dt} = -\gamma B_0$$

$$\phi(t) = -\gamma B_0 t$$