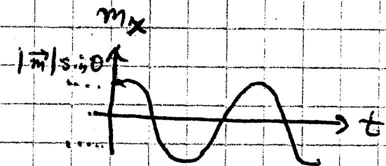
$$\hat{y}: -\gamma \cancel{\hbar} \cancel{\gamma} B_0 \cancel{\sin\theta \cos\phi} = \cancel{\hbar} \cancel{\gamma} \cancel{\sin\theta \cos\phi} \frac{d\phi}{dt}$$

$$\frac{d\phi}{dt} = -\gamma B_0 \Rightarrow \phi(t) = -\gamma B_0 t \quad \checkmark \text{ (check)}$$

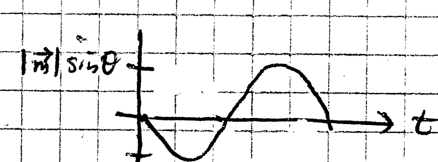
θ is fixed as $N_\theta = 0$

$$\Rightarrow \vec{m}(t) = |\vec{m}| \left[\sin\theta \cos\phi(t)\, \hat{x} + \sin\theta \sin\phi(t)\, \hat{y} + \cos\theta\, \hat{z} \right]$$

$$m_x(t) = |\vec{m}| \sin\theta \cos\phi(t) = |\vec{m}| \sin\theta \cos(\gamma B_0 t)$$

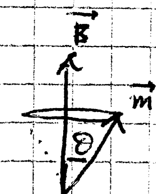


$$m_y(t) = |\vec{m}| \sin\theta \sin\phi(t) = -|\vec{m}| \sin\theta \sin(\gamma B_0 t)$$



precesses about $\hat{z}$ in left-hand (clockwise) sense

$$\vec{\omega}_{Larmor} = -\gamma B_0 \hat{z} \quad \text{independent of } \theta!!$$



Typical values: $\gamma_{proton} = 2.675 \times 10^8 \ \frac{\text{rad/s}}{\text{Tesla}}$

Earth Field: ~ 0.5 Gauss $\approx 5 \times 10^{-5}$ T

$$|\omega_L| = 13375 \ \frac{\text{rad}}{\text{s}} = 2\pi f \Rightarrow f = 2128 \text{ Hz}$$

Macroscopic Magnetostatics

classically,

- Atoms in matter have electrons in motion $\Rightarrow$ "currents" at atomic scale in matter $\Rightarrow$ magnetic moments!

Sketch of main concepts:

$$\vec{\nabla} \cdot \vec{B}_{micro} = 0 \longrightarrow \vec{\nabla} \cdot \underbrace{\vec{B}}_{\text{macroscopic}} = 0 \text{ (still; after averaging over microscopic)}$$

$\Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ still valid

Large number of atoms/molecules, each with own $\vec{m}_i$ magnetic moment

microscopic $\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle \quad N_i$: number density "bulk magnetization"

Macroscopic current density: from free flow of charge in the medium

So macroscopically, we have two different sources for a vector potential $\vec{A}$:

① macroscopic current $\vec{J}(\vec{x})$

② macroscopic magnetization $\vec{M}(\vec{x})$

\} + assuming no higher multipoles

$$\Rightarrow \vec{A}(\vec{x}) = \underbrace{\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'}_{①} + \underbrace{\frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{x}')\, d^3x' \times (\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}_{②}$$

just analog to microscopic dipole term $\frac{\mu_0}{4\pi}\frac{\vec{m}\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$ [see p. 107]

$$= \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \underbrace{\frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}\right]$$

Recall: $\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$

$$\Rightarrow \int d^3x'\, \vec{M}(\vec{x}')\times\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -\int d^3x'\, \vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\times\vec{M}(\vec{x}')$$

$$\left[\text{use: } -\nabla\psi\times\vec{a} = \psi\,\vec{\nabla}\times\vec{a} - \vec{\nabla}\times(\psi\vec{a})\right]$$

$$= \int d^3x'\, \frac{1}{|\vec{x}-\vec{x}'|}\,\vec{\nabla}_{x'}\times\vec{M}(\vec{x}') - \underbrace{\int d^3x'\, \vec{\nabla}_{x'}\times\left(\frac{1}{|\vec{x}-\vec{x}'|}\vec{M}(\vec{x}')\right)}_{\text{work on this } (=0)}$$

Look at $\hat{x}_j'$ - component:

$$\left[\vec{\nabla}_{x'}\times\left(\frac{1}{|\vec{x}-\vec{x}'|}\vec{M}(\vec{x}')\right)\right]_j = \vec{\nabla}x_j'\cdot\left[\vec{\nabla}_{x'}\times\left(\frac{1}{|\vec{x}-\vec{x}'|}\vec{M}(\vec{x}')\right)\right]$$

Then: by integration by parts:

$$\int_{V,\ \text{all space}} d^3x'\, \underbrace{\vec{\nabla}x_j'\cdot\left[\vec{\nabla}_{x'}\times\left(\frac{1}{|\vec{x}-\vec{x}'|}\vec{M}(\vec{x}')\right)\right]}_{(\vec{\nabla}u\cdot\vec{v})} = \overbrace{\oint_S x_j'\left[\vec{\nabla}_{x'}\times\left(\frac{1}{|\vec{x}-\vec{x}'|}\vec{M}(\vec{x}')\right)\right]\cdot\hat{n}\, da'}^{=0 \text{ at } \infty}$$

$$-\int_V d^3x'\, x_j'\left[\vec{\nabla}_{x'}\cdot\vec{\nabla}_{x'}\times(\ldots)\right]\Big\} = 0 \text{ as } \vec{\nabla}\cdot\vec{\nabla}\times\vec{v} = 0\ \forall\,\vec{v}$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{\nabla}_{x'} \times \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right]$$

[Compare to microscopic:

$$\vec{A}_{micro} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_{micro}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \Big]$$

recall: from bulk atoms'/molecules $\vec{m}_i$ magnetic moments

$\Rightarrow$ magnetization $\vec{M}$ contributes "effective current density" macroscopically:

$$\vec{J}_M \equiv \vec{\nabla} \times \vec{M}$$

$$\Rightarrow \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{[\vec{J}(\vec{x}') + \vec{J}_M(\vec{x}')]}{|\vec{x}-\vec{x}'|}$$

Just like $\vec{\nabla}\times\vec{B}_{micro} = \mu_0 \vec{J}_{micro}$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}(\vec{x}) + \vec{J}_M(\vec{x})\right) = \mu_0\left[\vec{J} + \vec{\nabla}\times\vec{M}\right]$$

$$\Rightarrow \vec{\nabla}\times(\vec{B} - \mu_0\vec{M}) = \mu_0\vec{J}$$

$$\vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)} = \vec{J}$$

when derive full ~~Maxwell equations~~ later need to modify to $\vec{\nabla}\times\vec{H} = \vec{J} + \partial_t\vec{D}$

Jackson: "the magnetic field"

we: "$\vec{H}$-field"

$$\left\{ \vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \quad \text{or} \Rightarrow \vec{B} = \mu_0\vec{H} + \mu_0\vec{M} \right. \Rightarrow \boxed{\begin{array}{l}\vec{\nabla}\times\vec{H} = \vec{J} \\ \vec{\nabla}\cdot\vec{B} = 0\end{array}}$$

Macroscopic Magnetostatics

in vacuum, $\vec{M} = 0$, $\vec{B} = \mu_0\vec{H}$

in medium/bulk, $\vec{B}$ augmented by $\vec{M}$

For isotropic diamagnetic and paramagnetic, $\vec{M} = \chi_m \vec{H}$ (linear relationship!)

χ_m: magnetic susceptibility (dimensionless)

magnetization that develops $\propto$ applied "$\vec{H}$" field

$\chi_m > 0$: paramagnets $\quad \chi_m < 0$: diamagnets

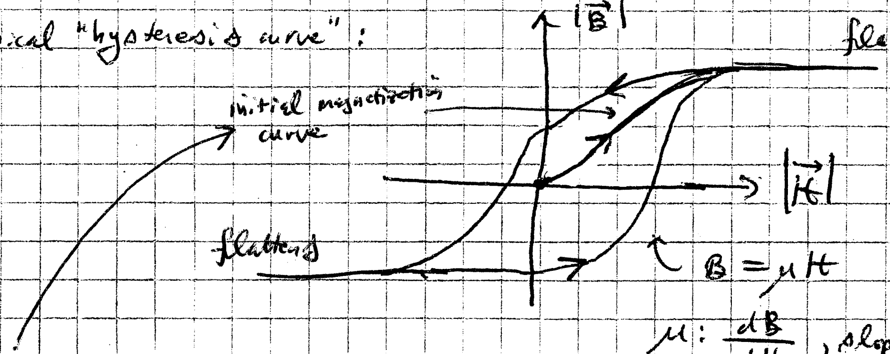
For such linear media: $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M} = \mu_0\vec{H} + \mu_0\chi_m\vec{H} \equiv \mu\vec{H}$

[$\frac{\mu}{\mu_0}$: "relative permeability"] $\quad \mu \equiv \mu_0(1+\chi_m)$; magnetic permeability

in vacuum: $\chi_m = 0$ (nothing to magnetize!), $\mu = \mu_0$

For ferromagnetic substances: non-linear, $\vec{B} = \vec{B}(\vec{H})$, highly non-linear

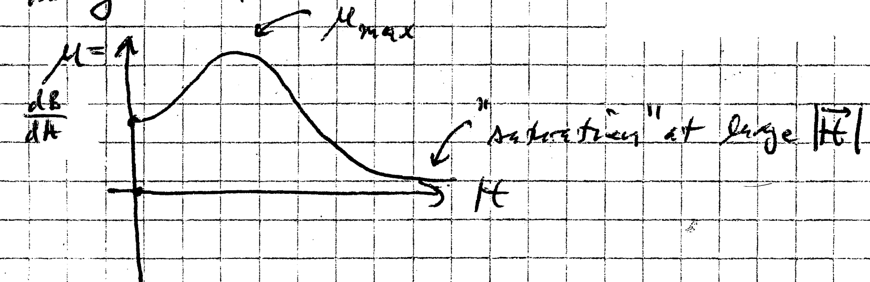
e.g., typical "hysteresis curve":



$B = \mu H$

μ: $\frac{dB}{dH}$, slope of $B(H)$ curve

"Permeability curve":



if start material at $(B, H) = (0, 0)$, e.g., if heat above Curie temperature in no magnetic field, then apply field H, traces out permeability curve. (along "initial magnetization curve")

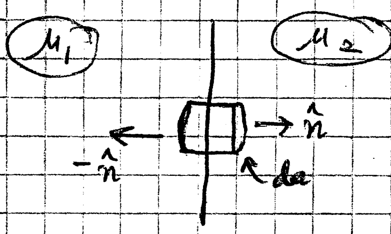
Typical ferromagnets, $\mu_{max} \sim 10^4 - 10^6$

<u>Boundary conditions</u>

$\mu_1 \gg \mu_2$	μ_2
$\vec{B}_1, \vec{H}_1$	$\vec{B}_2, \vec{H}_2$

For linear media, i.e., $\vec{B} = \mu\vec{H}$ (or, approximating ferromagnetic as linear)

- $\int_V (\vec{\nabla} \cdot \vec{B})\, d^3x = \oint_S \vec{B} \cdot \hat{n}\, da = 0$ (no free magnetic charge)



$$\int \left[\vec{B}_1 \cdot (-\hat{n}) + \vec{B}_2 \cdot (\hat{n})\right] da = 0$$

$$\Longrightarrow \quad \vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$$

Normal component of $\vec{B}$ is continuous! $\left[B_{1n} = B_{2n}\right]$

$\left[\text{Compare to: } (\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}\right]$