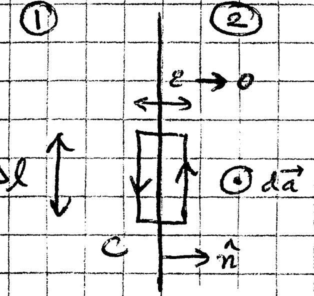
Because $\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$

$$\Rightarrow \mu_1 \vec{H}_1 \cdot \hat{n} = \mu_2 \vec{H}_2 \cdot \hat{n} \Rightarrow \vec{H}_2 \cdot \hat{n} = \frac{\mu_1}{\mu_2} \vec{H}_1 \cdot \hat{n}$$

$$\left[H_{2n} = \frac{\mu_1}{\mu_2} H_{1n} \right]$$

For now taking $\vec{\nabla} \times \vec{H} = \vec{J}$:

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{a} = \oint_C \vec{H} \cdot d\vec{l} \quad \text{[Stokes' Th'm]}$$



$$\Rightarrow \int_S \vec{J} \cdot d\vec{a} = \oint_C \vec{H} \cdot d\vec{l}$$

$[|\vec{K}|]$: $\frac{\text{current}}{\text{length}}$

If there is some surface current $\vec{K}$ flowing exactly on boundary surface, then:

$$\int_S \vec{J} \cdot d\vec{a} = (\vec{K} \cdot \hat{t}) \cdot \Delta l$$

(transverse unit vector) $\odot \hat{t}$ defined by C orientation (transverse unit vector)

$$\downarrow$$

$$\hat{n} \times \hat{t}$$

$$\oint_C \vec{H} \cdot d\vec{l} = \vec{H}_1 \cdot (\hat{n} \times \hat{t}) \cdot \Delta l - \vec{H}_2 \cdot (\hat{n} \times \hat{t}) \Delta l \quad *$$

$$= \left[\hat{t} \cdot (\vec{H}_1 \times \hat{n}) - \hat{t} \cdot (\vec{H}_2 \times \hat{n}) \right] \Delta l$$

[*using: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$]

$$\Rightarrow \vec{K} = \vec{H}_1 \times \hat{n} - \vec{H}_2 \times \hat{n}$$

$$= \hat{n} \times (\vec{H}_2 - \vec{H}_1)$$

if $\vec{K} \neq 0$, tangential component is discontinuous

If $\vec{K} = 0$: $\underbrace{\vec{H}_1 \times \hat{n} = \vec{H}_2 \times \hat{n}}_{\text{tangential component of } \vec{H} \text{ is continuous}}$

In Linear Media: $\frac{\vec{B}_1}{\mu_1} \times \hat{n} = \frac{\vec{B}_2}{\mu_2} \times \hat{n}$

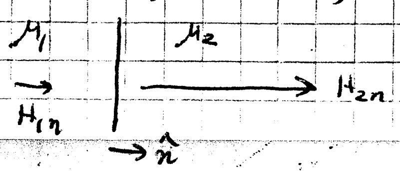
$$\vec{B}_2 \times \hat{n} = \frac{\mu_2}{\mu_1} \vec{B}_1$$

<u>Summary: $\vec{K} = 0$:</u>

$\vec{B}_1 \cdot \hat{n} = \vec{B}_2 \cdot \hat{n}$ $\qquad$ $\vec{B}_2 \times \hat{n} = \frac{\mu_2}{\mu_1} \vec{B}_1 \times \hat{n}$

$\frac{\mu_1}{\mu_2} H_1 \cdot \hat{n} = \hat{H}_2 \cdot \hat{n}$ $\qquad$ $\vec{H}_2 \times \hat{n} = \vec{H}_1 \times \hat{n}$

If $\mu_1 >> \mu_2$, $H_{2n} >> H_{1n}$, $\vec{H}_2 \times \hat{n} = \vec{H}_1 \times \hat{n}$ (tangential component)



→ Boundary conditions on $\tilde{\vec{H}}$ <u>same</u> as $\vec{E}$-field on surface of conductor, H_{2n} only in limit $\frac{\mu_2}{\mu_1} \to \infty$.

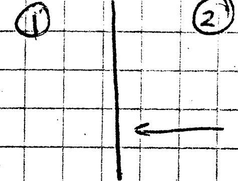
$[H_{2n} >> H_{1n}]$

$[H_{2t} = H_{1t}]$

under this assumption: $\mu_1 \gg \mu_2$;

$\Rightarrow$ can use electrostatic potential theory for solutions to $\vec{H}$!!

subject to boundary conditions



$\longleftarrow$ boundary is $\approx$ "equipotential" of Φ_M, $\vec{H}$ normal to these "equipotentials"

Methods for Solving Magnetostatic Boundary-Value Problems

$\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{H} = \vec{J}$ are starting point [for linear media: $\vec{B} = \mu\vec{H}$]

① "Vector Potential Method"

For linear media, $\vec{\nabla}\times\vec{H} = \vec{\nabla}\times\frac{1}{\mu}\vec{B} = \frac{1}{\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

using: $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{A})}_{=0 \text{ in Coulomb gauge}} - \vec{\nabla}^2\vec{A}$ (vector identity)

$\Rightarrow \quad -\vec{\nabla}^2\vec{A} = \mu\vec{J} \quad \Rightarrow$ Poisson Equation for each component:

$$-\vec{\nabla}^2 A_i = \mu J_i$$

② "Magnetic Scalar Potential Method"

If $\vec{J} = 0$ everywhere, $\vec{\nabla}\times\vec{H} = 0$. This coupled with $\vec{\nabla}\cdot\vec{B} = 0$, analogous to charge-free electrostatics $[\vec{\nabla}\times\vec{E} = 0,\ \vec{\nabla}\cdot\vec{E} = 0]$, so can write: $\vec{H} = -\vec{\nabla}\Phi_M$. $[\vec{E} = -\vec{\nabla}\Phi]$

Assuming linear media, $\vec{B} = \mu\vec{H}$, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot(\mu\vec{H}) = \mu\vec{\nabla}\cdot(-\vec{\nabla}\Phi_M) = 0$

$\Rightarrow \quad \vec{\nabla}^2\Phi_M = 0$ Laplace Equation

Solutions via the boundary conditions: (for linear media)

$\vec{B}_2\cdot\hat{n} = \vec{B}_1\cdot\hat{n} \qquad \vec{B}_2\times\hat{n} = \frac{\mu_2}{\mu_1}\vec{B}_1\times\hat{n}$

$\vec{H}_2\cdot\hat{n} = \frac{\mu_1}{\mu_2}\vec{H}_1\cdot\hat{n} \qquad H_2\times\hat{n} = \vec{H}_1\times\hat{n}$

③ "Hard Ferromagnet": $\vec{M}$ independent of applied/external fields (so $\vec{M}(\vec{x})$ just permanent/fixed at point $\vec{x}$)

For $\vec{J} = 0$: $\vec{\nabla}\cdot\vec{B} = \mu_0\vec{\nabla}\cdot(\vec{H}+\vec{M}) = 0$

$\Rightarrow$ using $\vec{H} = -\vec{\nabla}\Phi_M \Rightarrow \cancel{\mu_0}\vec{\nabla}\cdot(-\vec{\nabla}\Phi_M) + \cancel{\mu_0}\vec{\nabla}\cdot\vec{M} = 0 \Rightarrow +\vec{\nabla}^2\Phi_M = -\rho_M$

$\rho_M \equiv -\vec{\nabla}\cdot\vec{M}$

"effective magnetic-charge density"

$$\Rightarrow \quad \Phi_M(\vec{x}) = \frac{1}{4\pi}\int\frac{(-\vec{\nabla}'\cdot\vec{M}(\vec{x}'))\,d^3x'}{|\vec{x}-\vec{x}'|}$$

[just like: $\Phi = \frac{1}{4\pi\epsilon_0}\int\frac{d^3x'\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$]

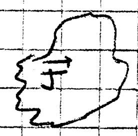
Example: $z=0$ **Mirror Currents** [Jackson Problem 5.17]

semi-infinite slab of material w/ $\mu_r = \frac{\mu}{\mu_0}$ for $z<0$

Some current distribution $\vec{J}(\vec{x})$ in $z>0$

$\mu/\mu_0 = 1$ (vacuum)

e.g.,



$\longrightarrow z$

$z<0$

(a) Show that for $z>0$ $\vec{B}$ can be calculated by replacing the material with an image / mirror current distribution of:

$$\left(\frac{\mu_r-1}{\mu_r+1}\right) J_x(x,y,-z)\,\hat{x} + \left(\frac{\mu_r-1}{\mu_r+1}\right) J_y(x,y,-z)\,\hat{y} + -\left(\frac{\mu_r-1}{\mu_r+1}\right) J_z(x,y,-z)\,\hat{z}$$

(b) Show that for $z<0$, $\vec{B}$ appears due to current distribution:

$$\frac{2\mu_r}{\mu_r+1}\vec{J} \text{ in medium of unit relative permeability.}$$

Solution: Define image currents:

$\vec{J_1}$: zero for $z>0$ (a)

$\vec{J_2}$: zero for $z<0$ (b)

to deal with these parts

The $\vec{B}$-field will then be of the form:

$$\underline{\underline{z>0}}: \quad \vec{B} = \frac{\mu_0}{4\pi}\int \frac{\left[\vec{J}(\vec{x}') + \vec{J_1}(\vec{x}')\right] \times (\vec{x}-\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|^3}$$

(treat $\vec{B}_{z>0}$ as due to the original current $\vec{J}$ in $z>0$ and image $\vec{J_1}$ entirely in $z<0$ region)

$$\underline{\underline{z<0}}: \quad \vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J_2}(\vec{x}') \times (\vec{x}-\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|^3}$$

($\vec{B}_{z<0}$ due to current of same form as original current $\vec{J}$ in $z>0$, but of modified strength, due to change in permeability across the boundary)

Normal component of $\vec{B}$ must be continuous ($\hat{z}$):

$$\Rightarrow B_z(x,y,0^+) = \frac{\mu_0}{4\pi}\int \frac{\left[(J_x + J_{1x})(y-y') - (J_y+J_{1y})(x-x')\right] d^3x'}{\left[(x-x')^2+(y-y')^2+(z')^2\right]^{3/2}}$$

$[\hat{z} = \hat{x}\times\hat{y}]$

and: $B_z(x,y,0-) = \frac{\mu_0}{4\pi}\int \frac{J_{2x}(y-y') - J_{2y}(x-x')}{[(x-x')^2+(y-y')^2+(z')^2]^{3/2}} d^3x'$

Enforcing the boundary condition gives:

$$\int d^3x' \left[(x-x')^2+(y-y')^2+(z')^2\right]^{-3/2} \cdot \left[((J_x+J_{1x})-J_{2x})(y-y') - ((J_y+J_{1y})-J_{2y})(x-x')\right] = 0$$

Have to be careful, since we defined: $\vec{J}_1 = 0$ for $z'>0$, $\vec{J}_2 = 0$ for $z'<0$ — i.e., $\vec{J}_1$ and $\vec{J}_2$ non-zero in different regions, z'.

But! Since $\int d^3x' (\ldots)$ only depends on z' via $(z')^2$, can take $z' \to -z'$ in $\vec{J}_1$ so that $\vec{J}$, $\vec{J}_1$, and $\vec{J}_2$ are all non-zero for $z'>0$.

$\Rightarrow$ To satisfy boundary condition, $\int (\ldots) d^3x'$ must vanish, or integrand $(\ldots) = 0$.

$$\Rightarrow \quad J_x(x,y,z) + J_{1x}(x,y,-z) - J_{2x}(x,y,z) = 0$$
$$J_y(x,y,z) + J_{1y}(x,y,-z) - J_{2y}(x,y,z) = 0$$

Now, boundary condition for tangential components: (assuming linear $\vec{B} = \mu\vec{H}$) (both $\hat{x}$ and $\hat{y}$) as $\vec{H}_1 \times \hat{n} = \vec{H}_2 \times \hat{n}$; $\hat{n} = \hat{z}$

with $z \leftrightarrow -z'$ for $\vec{J}_1$ terms:

$$H_y(x,y,0+) = \frac{1}{\mu_0} \cdot B_y(x,y,0+) = \frac{1}{4\pi}\int \frac{[(J_z+J_{1z})(x-x') - (J_x+J_{1x})(0 \mp z')]\, d^3x'}{[\ldots]^{3/2}}$$

$[\hat{y} = \hat{z}\times\hat{x}]$

$$H_y(x,y,0-) = \frac{1}{(\mu_0\mu_r)} \cdot B_y(x,y,0-) = \frac{1}{4\pi\mu_r}\int \frac{[J_{2z}(x-x') - J_{2x}(0-z')]\, d^3x'}{[\ldots]^{3/2}}$$

and: (similarly) with $z' \to -z'$ for $\vec{J}_1$ terms!

$$H_x(x,y,0+) = \frac{1}{4\pi}\int \frac{[(J_y+J_{1y})(0 \mp z') - (J_z+J_{1z})(y-y')]\, d^3x'}{[\ldots]^{3/2}}$$

$[\hat{x} = \hat{y}\times\hat{z}]$

$$H_x(x,y,0-) = \frac{1}{4\pi\mu_r}\int \frac{J_{2y}(0-z') - (J_{2z})(y-y')}{[\ldots]^{3/2}}$$

Enforcing the boundary conditions gives:

$\mu_r J_z(x,y,z) + \mu_r J_{1z}(x,y,-z) - J_{2z}(x,y,z) = 0$

$\mu_r J_x(x,y,z) - \mu_r J_{1x}(x,y,-z) - J_{2x}(x,y,z) = 0$

$\mu_r J_y(x,y,z) - \mu_r J_{1y}(x,y,-z) - J_{2y}(x,y,z) = 0$

$\mu_r J_z(x,y,z) + \mu_r J_{1z}(x,y,-z) - J_{2z}(x,y,z) = 0$

Now we have 5 equations for the boundary conditions, 6 unknowns: $\vec{J_1}$ and $\vec{J_2}$ in terms of $\vec{J}$

$J_x + J_{1x} - J_{2x} = 0$ $\qquad \mu_r J_z + \mu_r J_{1z} - J_{2z} = 0 \quad (*)$

$J_y + J_{1y} - J_{2y} = 0$ $\qquad \mu_r J_x - \mu_r J_{1x} - J_{2x} = 0$

$\mu_r J_y - \mu_r J_{1y} - J_{2y} = 0$

$\mu_r J_z + \mu_r J_{1z} - J_{2z} = 0 \quad (*)$ same!

$$\mu_r J_y - \mu_r J_{1y} - J_y - J_{1y} = 0 \Rightarrow J_y(\mu_r - 1) = J_{1y}(\mu_r + 1)$$

$$J_{1y} = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y$$

or restoring "indices"

$$\underline{\underline{J_{1y}(x,y,-z) = J_y(x,y,z)\cdot\left(\frac{\mu_r-1}{\mu_r+1}\right)}}$$

$$\mu_r J_x - \mu_r J_{1x} - J_x - J_{1x} = 0$$

$$J_x(\mu_r - 1) = J_{1x}(\mu_r + 1) \Rightarrow J_{1x} = \left(\frac{\mu_r-1}{\mu_r+1}\right) J_x$$

$$\underline{\underline{J_{1x}(x,y,-z) = J_x(x,y,z)\left(\frac{\mu_r-1}{\mu_r+1}\right)}}$$

$\mu_r J_x - \mu_r(J_{2x} - J_x) = J_{2x}$

$J_x(\mu_r + \mu_r) = J_{2x}(\mu_r + 1)$

$$\underline{\underline{J_{2x}(x,y,z) = \frac{2\mu_r}{\mu_r+1} J_x(x,y,z)}}$$

$\mu_r J_y - \mu_r(J_{2y} - J_y) - J_{2y} = 0$

$\mu_r J_y - \mu_r J_{2y} + \mu_r J_y - J_{2y} = 0$

$$\underline{\underline{J_{2y}(x,y,z) = \frac{2\mu_r J_y(x,y,z)}{\mu_r+1}}}$$

To find J_{1z} and J_{2z}, need another equation,

For magnetostatics: $\vec{\nabla}\cdot\vec{J_2} = 0$

$$\Rightarrow \frac{\partial}{\partial x} J_{2x} + \partial_y J_{2y} + \partial_z J_{2z}$$

$$= \partial_x\left(\frac{2\mu_r}{\mu_r+1}\right)J_x + \partial_y\left(\frac{2\mu_r J_y}{\mu_r+1}\right) + \partial_z\left(\mu_r J_z + \mu_r J_{1z}\right)$$

$$= \frac{2\mu_r}{\mu_r+1}\left(\partial_x J_x + \partial_y J_y\right) + \mu_r \partial_z J_z + \mu_r \partial_z J_{1z}$$

$$\Rightarrow \frac{2\mu_r}{\mu_r+1}\left(\partial_x J_x + \partial_y J_y + \tfrac{1}{2}(\mu_r+1)\partial_z J_z\right) + \mu_r \partial_z J_{1z} = 0$$

$$\frac{2\mu_r}{\mu_r+1}\left(\underbrace{\partial_x J_x + \partial_y J_y + \partial_z J_z}_{=0}\right) + \frac{2\mu_r}{\mu_r+1}\left(\tfrac{1}{2}(\mu_r+1)\partial_z J_z - \partial_z J_z\right) + \mu_r \partial_z J_{1z} = 0$$

$$\mu_r \partial_z J_z - \frac{2\mu_r}{\mu_r+1}\partial_z J_z + \mu_r \partial_z J_{1z} = 0$$

$$\Rightarrow \partial_z\left(\frac{2\mu_r}{\mu_r+1} J_z - \mu_r J_z\right) = \partial_z\left(\mu_r J_{1z}\right)$$

$$\left(\frac{2\mu_r}{\mu_r+1} - \mu_r\right) J_z = \mu_r J_{1z}$$

$$\left(\frac{2}{\mu_r+1} - 1\right) J_z = J_{1z}$$

$$J_{1z} = \frac{1-\mu_r}{\mu_r+1} J_z = -\frac{\mu_r-1}{\mu_r+1} J_z$$

$$\Rightarrow \quad J_{1z}(x,y,-z) = -\frac{\mu_r-1}{\mu_r+1} J_z(x,y,z) \quad \checkmark$$

$$J_{2z} = \mu_r J_z + \mu_r J_{1z}$$

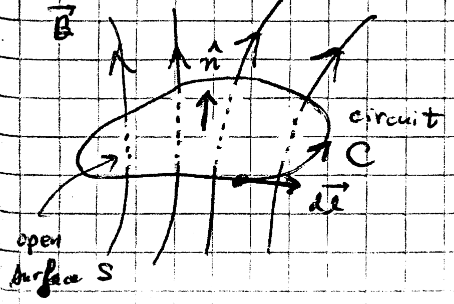
$$= \mu_r J_z + -\frac{\mu_r(\mu_r-1)}{\mu_r+1} J_z$$

$$= \frac{\mu_r^2 + \mu_r - \mu_r^2 + \mu_r}{\mu_r+1} J_z = \frac{2\mu_r}{\mu_r+1} J_z$$

$$\Rightarrow \quad J_{2z}(x,y,z) = \frac{2\mu_r}{\mu_r+1} J_z(x,y,z) \quad \checkmark$$

Faraday's Law of Induction

So far, static currents $\Rightarrow$ static magnetic fields.



Magnetic flux threading C is:

$$F = \int_S \vec{B}\cdot\hat{n}\,da \qquad \text{(in lab frame)}$$

Electromotive force around the circuit is:

$$\mathcal{E} = \oint_C \vec{E}'\cdot\vec{dl} \qquad \text{(EMF)}$$

$\vec{E}'$: electric field at element $\vec{dl}$ of C in coordinate system/frame for which $\vec{dl}$ is at rest $\Rightarrow$ causes force $= q\vec{E}'$ on charged particle in that frame