Faraday's observations: $\mathcal{E} = -k \dfrac{dF}{dt}$, As see k depends on choice of units, i.e., ~~not~~ an empirical constant ⊛

negative sign due to Lenz's Law, induced current, and accompanying magnetic flux, opposes change of F

$[k = 1 \text{ in SI}, \quad k = \frac{1}{c} \text{ in Gaussian (cgs)}]$

$\vec{E}'$ is electric field in coordinate system/frame for which $d\vec{l}$ is at rest

$$\Rightarrow \oint_C \vec{E}' \cdot d\vec{l} = -k \frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, da$$

$\Rightarrow$ Faraday's Law $\Rightarrow$ consider consequences of Galilean invariance

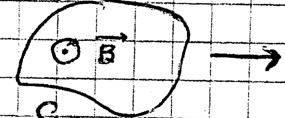
force $= q\vec{E}'$ on charge (i.e., causes current to flow)

flux can be changed by changing $\vec{B}$ or changing the shape/orientation/position of the circuit C in $\vec{B}$-field

applies to any geometric path C (not necessarily a "circuit")

$\Longrightarrow$ implies relation between $\vec{B}$- and $\vec{E}$-fields themselves

If circuit C moving with constant velocity $\vec{v}$ in some direction:



($\vec{v}$ take constant for now)

$$\frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\, da = \int_S \frac{d\vec{B}}{dt} \cdot \hat{n}\, da$$

$$\vec{B} = \vec{B}(\vec{x}(t), t)$$

flux changes in time at a point; translation of circuit (boundary C) through non-uniform field

$$\text{e.g.,} \quad \frac{dB_x}{dt} = \frac{\partial B_x}{\partial t} + \frac{\partial B_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial B_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial B_x}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial B_x}{\partial t} + (\vec{v} \cdot \vec{\nabla}) B_x$$

$$\Rightarrow \frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{B}$$

by vector identity

$$= \frac{\partial \vec{B}}{\partial t} + \Big[\underbrace{\vec{v}(\vec{\nabla}\cdot\vec{B})}_{=0} - \underbrace{\vec{B}(\vec{\nabla}\cdot\vec{v})}_{=0 \text{ for}} + \underbrace{(\vec{B}\cdot\vec{\nabla})\vec{v}}_{=0} - \vec{\nabla}\times(\vec{v}\times\vec{B})\Big]$$

fixed/constant $\vec{v}$

$$= \frac{\partial \vec{B}}{\partial t} + -\vec{\nabla}\times(\vec{v}\times\vec{B})$$

$$\Rightarrow \int_S \frac{d\vec{B}}{dt}\cdot\hat{n}\, da = \int_S \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}\times\vec{B})\right]\cdot\hat{n}\, da$$

$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da + -\int_S \left[\vec{\nabla}\times(\vec{v}\times\vec{B})\right]\cdot\hat{n}\, da$$

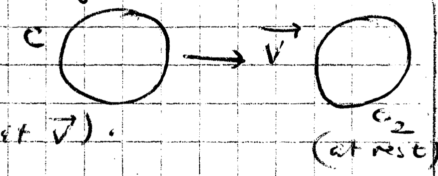
Stokes' Thm

$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da - \oint_C (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$\Longrightarrow \oint_C \left[\vec{E}' - k(\vec{v}\times\vec{B})\right]\cdot d\vec{l} = -k\int \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da$$

$\vec{E}'$ in frame affixed to a moving $d\vec{l}$ (i.e., the electric field in the frame where $d\vec{l}$ at rest, coordinate system moving at $\vec{v}$ w.r.t. lab frame)

This result assumed the circuit C was <u>moving</u> with (constant) velocity $\vec{V}$.



Consider a second, identical circuit, C_2, which is at <u>rest</u> in the lab frame (i.e., w.r.t. C moving at $\vec{V}$).

~~Suppose that at some time the two~~ identical ~~circuits~~ C and C_2 are (hypothetically) at exactly the same point in space (i.e., they overlap exactly).

In this circuit C_2 at <u>rest</u>, Faraday's law will then be:

now $\vec{E}$-field in laboratory frame

$$\oint_C \vec{E}\cdot d\vec{l} = -k\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da \qquad [\text{NO translation terms!!}]$$

from moving through non-uniform field!

By Galilean invariance, physics must be the same! (i.e., EMF <u>must</u> be the same, or induced current, must be the same at this instant in time)

$$\vec{E}' - k(\vec{V}\times\vec{B}) = \vec{E}$$

field in frame moving at velocity $\vec{V}$ — field in lab frame

$\Rightarrow$ later, PHY 613 Lorentz invariance (special relativity)

$$\text{or} \quad \vec{E}' = \vec{E} + k(\vec{V}\times\vec{B})$$

To determine k: Consider a charge q in some reference frame moving at $\vec{V}$ w.r.t. lab frame. Initially release from rest.

At this instant in time (@ release):

To observer in lab frame sees charge q moving at $\vec{V}$ subjected to electric field $\vec{E}$ and $\vec{B}$

$$\Rightarrow \vec{F} = q\vec{E} + q\vec{V}\times\vec{B}$$

To observer fixed in <u>moving</u> reference frame, sees q subjected to only $\vec{E}'$ (no relative motion in this frame!)

$$\vec{F} = q\vec{E}' \overset{\text{must}}{\Rightarrow} \vec{E}' = \vec{E} + \vec{V}\times\vec{B} \Rightarrow k = 1 \text{ in SI}$$

$$\Rightarrow \underline{\underline{\vec{E}' = \vec{E} + (\vec{V}\times\vec{B})}} \qquad [\text{SI units}]$$

<u>Summary of Faraday's Law:</u>

electric field in coordinate frame moving with velocity $\vec{V}$ relative to lab frame

$$\oint_C \vec{E}'\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\, da \implies \text{changing magnetic flux generates electric field!}$$

Note:

$$\Rightarrow \oint_C [\vec{E} + (\vec{V}\times\vec{B})]\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da + \int_S [\vec{\nabla}\times(\vec{V}\times\vec{B})]\cdot\hat{n}\, da$$

by Stoke's Law

$$\Rightarrow \int_S [\vec{\nabla}\times\vec{E} + \cancel{\vec{\nabla}\times(\vec{V}\times\vec{B})}]\cdot\hat{n}\, da = \int_S \left[-\frac{\partial\vec{B}}{\partial t} + \cancel{\vec{\nabla}\times(\vec{V}\times\vec{B})}\right]\cdot\hat{n}\, da$$

$$\Longrightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Time-dependent generalization of $\vec{\nabla}\times\vec{E} = 0$.
Differential form of Faraday's Law.

Magnetic Field Energy

Assuming a linear relationship between $\vec{B}$ and $\vec{H}$ ($\vec{B}=\mu\vec{H}$), total magnetic energy is:

$$W=\frac{1}{2}\int \vec{H}\cdot\vec{B}\,d^3x$$

Recall: energy in field is total work required to establish the field (starting from zero)

(⇒ establishing a current, $0 \to I$, requires work against EMF induced, which opposes current, by Lenz's Law)

Note: if $\mu=\mu_0$, $\vec{B}=\mu_0\vec{H} \Rightarrow W=\frac{1}{2}\frac{1}{\mu_0}\int \vec{B}^2\,d^3x$

Also, $W=\frac{1}{2}\int \vec{J}\cdot\vec{A}\,d^3x$ interaction energy of current $\vec{J}$ in vector potential $\vec{A}$ (including self-contributions)

As:
$$W=\frac{1}{2}\int \vec{H}\cdot(\vec{\nabla}\times\vec{A})\,d^3x$$
$$=\frac{1}{2}\int\left[\vec{\nabla}\cdot(\vec{H}\times\vec{A})+\vec{A}\cdot(\vec{\nabla}\times\vec{H})\right]d^3x$$
$$=\underbrace{\frac{1}{2}\oint_{S\text{ at }\infty}(\vec{H}\times\vec{A})\cdot d\vec{a}}_{=0\text{ for localized }\vec{H},\vec{A}}+\underbrace{\frac{1}{2}\int\vec{A}\cdot\vec{J}\,d^3x}_{=\frac{1}{2}\int d^3x\,\vec{J}\cdot\vec{A}\ \checkmark}$$

e.g., if two sources of $\vec{J}$, w/ factor of $\frac{1}{2}$, integral over both current densities in total $\vec{A}$

$$W=\frac{1}{2}\int(\vec{J}_1+\vec{J}_2)\cdot(\vec{A}_1+\vec{A}_2)\,d^3x$$

Coefficients of Self- and Mutual-Inductance

Just like we defined coefficients of capacitance, where we write potential energy of a system of n conductors,

$$W_E=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}V_iV_j$$

we define coefficients of mutual and self-inductance for a system of N distinct circuits, each with current I_i, as:

$$W=\frac{1}{2}\sum_{i=1}^{N}L_iI_i^2+\sum_{i=1}^{N}\sum_{j>i}M_{ij}I_iI_j$$

because in general:

$$W=\frac{1}{2}\int d^3x\,\vec{J}\cdot\vec{A}\quad\text{and}\quad\vec{J}\propto I,\ \text{and}\ \vec{A}\propto\int\frac{\vec{J}\,d^3x'}{|\vec{x}-\vec{x}'|}\propto I$$

with the integrals $\int d^3x$ dependent on the geometry

$\Longrightarrow$ L_i and M_{ij} depend, generally, on the geometry of the circuits / currents and current distribution

Rigorously:

$$W=\frac{1}{2}\int d^3x\,\vec{J}(\vec{x})\cdot\vec{A}(\vec{x})\qquad\vec{A}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{d^3x'\,\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$=\frac{1}{2}\int d^3x\,\vec{J}(\vec{x})\cdot\frac{\mu_0}{4\pi}\int d^3x'\,\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}=\frac{\mu_0}{8\pi}\int d^3x\int d^3x'\,\frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$