Now, for the problem of N circuits, we can break up the integral into a sum of integrals:

$$W = \frac{\mu_0}{8\pi} \sum_{i=1}^{N} \int d^3x_i \sum_{j=1}^{N} \int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|}$$

over each circuit i: $\int d^3x_i \vec{J}_i$ w/ $\vec{A}_j$ due to circuit $\int d^3x_j \vec{J}_j$

These sums are unrestricted, and we define the $i=j$ terms to be:

$$W_{ii} = \frac{\mu_0}{8\pi} \int_{C_i} d^3x_i \int_{C_i} d^3x_i' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_i(\vec{x_i}')}{|\vec{x_i} - \vec{x_i}'|} \equiv \frac{1}{2} \underbrace{\frac{\mu_0}{4\pi I_i^2} \int\int (\ldots)}_{\equiv L_i} \cdot I_i^2$$

$$\Rightarrow L_i \equiv \frac{\mu_0}{4\pi I_i^2} \int d^3x_i \int d^3x_i' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_i(\vec{x_i}')}{|\vec{x_i} - \vec{x_i}'|}$$

self-inductance of circuit i (related to magnetic energy required to establish current in a circuit)

high self inductance ⇒ large amount of energy required to establish current (inductor in circuit opposes current flow)

For the $i \neq j$ terms:

$$\frac{\mu_0}{8\pi} \sum_{i=1}^{N} \int d^3x_i \sum_{j=1}^{N} \int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|}$$

This is symmetric in i and j, so we can define instead a restricted sum: (under $i \leftrightarrow j$)

$$\frac{\mu_0}{8\pi} \left[\sum_{i=1}^{N} \int d^3x_i \sum_{j=1}^{N} \int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|} \right]$$

$$= \frac{\mu_0}{8\pi} \left[2 \sum_{i=1}^{N} \int d^3x_i \sum_{j>i}^{N} \int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|} \right]$$

$$= \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \frac{1}{I_i} \int d^3x_i \sum_{j>i}^{N} \frac{1}{I_j} \int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|} (I_i I_j)$$

$$\equiv \sum_{i=1}^{N} \sum_{j>i}^{N} I_i I_j \, M_{ij}$$

$$M_{ij} \equiv \frac{\mu_0}{4\pi} \frac{1}{I_i I_j} \int d^3x_i \left(\int d^3x_j' \frac{\vec{J}_i(\vec{x_i}) \cdot \vec{J}_j(\vec{x_j}')}{|\vec{x_i} - \vec{x_j}'|} \right) = \frac{1}{I_i I_j} \underbrace{\int d^3x_i \, \vec{J}_i(\vec{x}) \cdot \vec{A}_j(\vec{x}_i)}_{= W_{ij}}$$

mutual inductance of circuits i and j (related to magnetic interaction energy of two currents) or current $\vec{J}_i$ in field due to vector potential $\vec{A}_j$ — related to energy needed to establish current in circuit i due to interaction w/ circuit j

Quasi-static Magnetic Fields

Faraday's Law: if $\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, da \neq 0$, electric field is created

"Quasi-static": if time variation not too rapid, $\vec{B}$-fields dominate. Valid for regime in which speed of light can be neglected, and fields treated as if they propagated instantaneously.

Maxwell Equations

So far, we have for the basic laws of electromagnetism:

$\vec\nabla\cdot\vec D=\rho$ "Coulomb's Law"

$\underbrace{\vec\nabla\times\vec H=\vec J}_{\text{special case: derived for}}$ but! special case: derived for $\vec\nabla\cdot\vec J=0$! (ρ = const) "Ampere's Law" $\Rightarrow$ Note: $\overbrace{\vec\nabla\cdot\vec J=0}^{\text{by assumption}}$ and $\vec\nabla\cdot\vec\nabla\times\vec H=0$ is also satisfied!

$\vec\nabla\times\vec E=-\partial_t\vec B$ "Faraday's Law"

$\vec\nabla\cdot\vec B=0$ no magnetic monopoles

All derived for static fields, except for Faraday's law.

Recall: continuity equation for charge:

$$\vec\nabla\cdot\vec J+\partial_t\rho=0$$

But, note we can write: $\vec\nabla\cdot\vec J+\partial_t\rho=\vec\nabla\cdot\left(\vec J+\partial_t\vec D\right)=0$ $\quad\rho=\vec\nabla\cdot\vec D$

So if we replace: $\vec J\to\vec J+\partial_t\vec D$ in Ampere's law, we have:

$\vec\nabla\times\vec H=\vec J+\frac{\partial\vec D}{\partial t}$ and still satisfy the general vector identity that $\vec\nabla\cdot(\vec\nabla\times\vec V)=0$ $\forall\vec V$

$\vec\nabla\cdot(\vec\nabla\times\vec H)=0$ and $\vec\nabla\cdot(\vec J+\partial_t\vec D)=\vec\nabla\cdot\vec J+\partial_t\rho=0$ ✓

Note: if electric field $\vec D$ is static, just get result from before: $\vec\nabla\times\vec H=\vec J$

"Displacement Current": $\partial_t\vec D$ $\Longrightarrow$ changing electric field causes a magnetic field, even if no current, $\vec J=0$!

[converse of Faraday's law, a "symmetry" between $\vec E$ and $\vec B$] leads to EM waves, radiation, etc.

Complete set of "Maxwell Equations"

$$\vec\nabla\cdot\vec D=\rho\qquad\vec\nabla\times\vec H=\vec J+\partial_t\vec D$$

$$\vec\nabla\cdot\vec B=0\qquad\vec\nabla\times\vec E=-\partial_t\vec B$$

Maxwell Equations in Terms of Vector and Scalar Potentials and Gauge Transformations

As we know, since $\vec\nabla\cdot\vec B=0$, can write $\vec B=\vec\nabla\times\vec A$

Thus, Faraday's law is: $\vec\nabla\times\vec E+\partial_t\vec B=\vec\nabla\times\vec E+\partial_t\vec\nabla\times\vec A=\underbrace{\vec\nabla\times(\vec E+\partial_t\vec A)}_{\text{can write as }\vec\nabla(\text{scalar})}=0$

Can write $\vec E+\partial_t\vec A=-\vec\nabla\Phi\Rightarrow\vec E=-\vec\nabla\Phi-\partial_t\vec A$ $\quad$ as $\vec\nabla\times(\vec\nabla(\text{scalar}))=0$

In vacuum, $\vec{D} = \varepsilon_0 \vec{E}$, $\vec{H} = \frac{\vec{B}}{\mu_0}$, so we can write:

$$\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\varepsilon_0\vec{E} = \vec{\nabla}\cdot\varepsilon_0\left(-\vec{\nabla}\Phi - \partial_t\vec{A}\right) = \rho$$

$$\Rightarrow \boxed{\vec{\nabla}^2\Phi + \partial_t(\vec{\nabla}\cdot\vec{A}) = -\rho/\varepsilon_0}$$

$$\vec{\nabla}\times\vec{H} = \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right) = \frac{1}{\mu_0}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{1}{\mu_0}\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}\right];$$

$$= \vec{J} + \partial_t\vec{D} = \vec{J} + \partial_t\varepsilon_0\vec{E} = \vec{J} + \varepsilon_0\partial_t\left(-\vec{\nabla}\Phi - \partial_t\vec{A}\right)$$

$$= \vec{J} - \varepsilon_0\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) - \varepsilon_0\partial_t^2\vec{A}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0\vec{J} - \mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\left[\text{Recall: } c^2 = \frac{1}{\varepsilon_0\mu_0}\right]$$

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\vec{J}} \qquad (*)$$

Recall: $\vec{B}$ invariant under "gauge transformation"

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \quad \Lambda\text{: scalar function}, \quad \text{as } \vec{\nabla}\times(\vec{A}+\vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A} \text{ as } \vec{\nabla}\times\vec{\nabla}\Lambda = 0$$

But! If we do this, we have:

$$\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A}' = -\vec{\nabla}\Phi - \partial_t\vec{A} - \partial_t\vec{\nabla}\Lambda = -\vec{\nabla}(\Phi + \partial_t\Lambda) - \partial_t\vec{A}$$

Need: $\Phi \to \Phi' = \Phi - \partial_t\Lambda$ for $\vec{E}$ to be invariant

$\Rightarrow$ Under such a gauge transformation, $\vec{B}$ and $\vec{E}$ are invariant

Now, under such a gauge transformation: $(*)$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} \to \vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\Phi'}{\partial t}$$

$$= \vec{\nabla}\cdot(\vec{A} + \vec{\nabla}\Lambda) + \frac{1}{c^2}\partial_t(\Phi - \partial_t\Lambda)$$

$$= \vec{\nabla}\cdot\vec{A} + \vec{\nabla}^2\Lambda + \frac{1}{c^2}\partial_t\Phi - \frac{1}{c^2}\partial_t^2\Lambda \Rightarrow \vec{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right)$$

If the gauge function Λ satisfies a "restricted gauge transformation":

$$\vec{\nabla}^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0 \Rightarrow \underline{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\partial_t\Phi = 0} \quad \text{"Lorenz condition"}$$