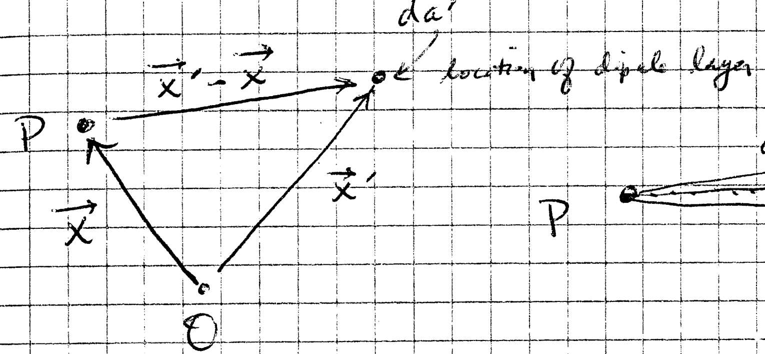
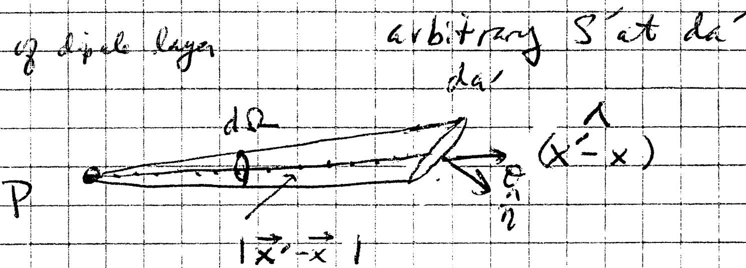
Back to general case:

(11)

$$\Phi_P(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_{S'} D(\vec{x}')\,\underbrace{\hat{n}\cdot\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da'}$$





$$\hat{n}\cdot\vec{\nabla}_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)da' = \hat{n}\cdot\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}da' = -\frac{(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3}\cdot\hat{n}\,da'$$

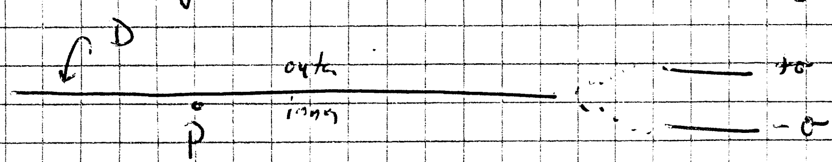
$$= -\frac{\cos\theta}{|\vec{x}'-\vec{x}|^2}da' = -\,d\Omega$$

Note: (sign convention)
- if viewing "inner" surface of da' from P, $d\Omega > 0$
- outer, $d\Omega < 0$

$$\Rightarrow \quad \Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int_S D(\vec{x}')\,d\Omega$$

If $D(\vec{x}')$ constant $= D$ $\Rightarrow$ $\Phi(\vec{x}) = -\frac{D}{4\pi\epsilon_0}\cdot$(solid angle subtended by point P)

So upon crossing some surface with constant dipole moment density D:



Put P infinitely close (infinitesimal distance away from surface):

$$\Phi(\text{inner}) = -\frac{D}{4\pi\epsilon_0}(2\pi) = -\frac{D}{2\epsilon_0}$$

$$\Phi(\text{outer}) = -\frac{D}{4\pi\epsilon_0}(-2\pi) = +\frac{D}{2\epsilon_0}$$

$\Rightarrow$ upon crossing the surface,

$$\Phi(\text{outer}) - \Phi(\text{inner}) = \frac{D}{\epsilon_0}$$

$\Rightarrow$ discontinuity in Φ across dipole-moment surface density

$\Rightarrow$ Analogous to discontinuity in $\vec{E}$ across a surface charge density
(normal component)

We showed earlier that: (for an electrostatic field)

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = 0 \iff \vec{E} = -\vec{\nabla}(\text{scalar function})$$

$$= -\vec{\nabla}\Phi \quad (\text{now call "}\Phi\text{"})$$

"the scalar potential"

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\Phi) = \partial_i(-\partial_i\Phi) = -\partial_i\partial_i\Phi$$

$$\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(-\frac{\partial\Phi}{\partial x}\hat{x} + -\frac{\partial\Phi}{\partial y}\hat{y} + -\frac{\partial\Phi}{\partial z}\hat{z}\right)$$

$$= -\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial z^2} = -\vec{\nabla}^2\Phi$$

rectangular: $f(x, y, z)$

$$\Rightarrow \boxed{\vec{\nabla}^2\Phi = -\frac{\rho}{\varepsilon_0}} \qquad \text{"Poisson equation"} \qquad \left[\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right]$$

In regions of space where $\rho = 0$ (i.e., no charge)

$$\boxed{\vec{\nabla}^2\Phi = 0} \qquad \text{"Laplace equation"}$$

We already showed previously that we can write a solution for $\Phi(\vec{x})$ as:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')\, d^3x'}{|\vec{x}-\vec{x}'|} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')\, d^3x'}{\sqrt{(\vec{x}-\vec{x}')^2}}$$

$= (\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')$

[good mathematical exercise]

Let's prove this satisfies the Poisson equation! But note this is singular (blows up!) at $\vec{x}' = \vec{x}$!

so let's use a common trick, and define

$$\Phi(\vec{x}) = \lim_{a\to 0}\Phi_a(\vec{x}) = \lim_{a\to 0}\left[\frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')\, d^3x'}{\sqrt{(\vec{x}-\vec{x}')^2 + a^2}}\, d^3x'\right]$$

$$\Rightarrow \quad \vec{\nabla}^2\Phi(\vec{x}) = \lim_{a\to 0}\left[\frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\,\vec{\nabla}^2\frac{1}{[r^2+a^2]^{1/2}}\, d^3x'\right]$$

[w.r.t. $\vec{x}$, so can take inside integral $\qquad r \equiv |\vec{x}-\vec{x}'|$